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P-TIME decidability of NL1 with assumptions

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Abstract

Buszkowski (2005) showed that systems of Non-associative Lambek Calculus with finitely many non-logical axioms are decidable in polynomial time and generate context-free languages. The same holds for systems with unary modalities, studied in Moortgat (1997), *n*-ary operations, and the rule of permutation, studied in Jäger (2004). The polynomial time decidability for Classical Non-associative Lambek Calculus was established by de Groote and Lamarche (2002). We study Non-associative Lambek Calculus with identity enriched with a finite set of assumptions. To prove that this system is decidable in polynomial time we adapt the method used in Buszkowski (2005). The context-freeness of the languages generated of the systems of Non-associative Lambek Calculus is also established.

Keywords LAMBEK CALCULUS, P-TIME DECIDABILITY

5.1 Introduction and preliminaries

Non-logical axioms can be of interest for linguistics for several reason. We can use them to describe subcategorization in natural language. For instance, restrictive adjectives are a sub-category of adjectives. Further, by enriching Non-associative Lambek Calculus with finitely new axioms, we can improve its expressibility without lacking the nice computational simplicity.

First we describe the formalism of Non-associative Lambek Calculus with identity constant (NL1). Let At = {p, q, r, . . .} be the denumerable set of atoms (primitive types).

The set of formulas (also called types) Tp1 is defined as the smallest set fulfilling the following conditions:

- $\mathbf{1} \in$ Tp1,

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- At $\subseteq$ Tp1,
- if $A, B \in$ Tp1, then $(A \bullet B) \in$ Tp1, $(A/B) \in$ Tp1, $(A \backslash B) \in$ Tp1, where binary connectives $\backslash$, $/$, $\bullet$, are called *left residuation*, *right residuation*, and *product*, respectively.

The set of formula structures STR1 is defined recursively as follows:

- $\Lambda \in$ STR1, where Λ denotes an empty structure,
- Tp1 $\subseteq$ STR1; these formula structures are called atomic formula structures,
- if $X, Y \in$ STR1, then $(X \circ Y) \in$ STR1.

We set $(X \circ \Lambda) = (\Lambda \circ X) = X$.

Substructures of a formula structure are defined in the following way:

- Λ is the only substructure of Λ,
- if X is an atomic formula structure, then Λ and X are the only substructures of X,
- if $X = (X_1 \circ X_2)$, then X and all substructures of X_1 and X_2 are substructures of X.

By $X[Y]$ we denote a formula structure X with a distinguished substructure Y, and by $X[Z]$ - the substitution of Z for Y in X.

Sequents are formal expressions $X \to A$ such that $A \in$ Tp1, $X \in$ STR1.

The Gentzen-style axiomatization of the calculus NL1 employs the axiom schemas:

$$\text{(Id)} \quad A \to A \qquad\qquad (\mathbf{1}\text{R}) \quad \Lambda \to \mathbf{1}$$

and the following rules of inference:

$$(\mathbf{1}\text{L}) \quad \frac{X[\Lambda] \to A}{X[\mathbf{1}] \to A},$$

$$(\bullet\text{L}) \quad \frac{X[A \circ B] \to C}{X[A \bullet B] \to C}, \qquad (\bullet\text{R}) \quad \frac{X \to A; \quad Y \to B}{X \circ Y \to A \bullet B},$$

$$(\backslash\text{L}) \quad \frac{Y \to A; \quad X[B] \to C}{X[Y \circ (A \backslash B)] \to C}, \qquad (\backslash\text{R}) \quad \frac{A \circ X \to B}{X \to A \backslash B},$$

$$(/\text{L}) \quad \frac{X[A] \to C; \quad Y \to B}{X[(B/A) \circ Y] \to C}, \qquad (/\text{R}) \quad \frac{X \circ B \to A}{X \to A/B},$$

$$\text{(CUT)} \quad \frac{Y \to A; \quad X[A] \to B}{X[Y] \to B}.$$

For any system S we write S $\vdash X \to A$ if the sequent $X \to A$ is derivable in S.

The most general models of NL1 are residuated groupoid with identity.

A *residuated groupoid* with identity is a structure

$$\mathcal{M} = (M, \leq, \cdot, \backslash, /, 1)$$

such that

- $(M, \cdot, 1)$ is a groupoid with identity in which $a \cdot 1 = a$, $1 \cdot a = a$ for all $a \in M$,
- $(M, \leq)$ is a poset ,
- $\backslash, /$ are binary operations on M satisfying the equivalences :

$$\text{(RES)} \quad ab \leq c \quad \text{iff} \quad b \leq a\backslash c \quad \text{iff} \quad a \leq c/b$$

for all $a, b, c \in M$.

Every residuated groupoid fulfills the following monotonicity laws:

$$\text{(MON)} \quad \text{If} \quad a \leq b \quad \text{then} \quad ca \leq cb \quad \text{and} \quad ac \leq bc$$

$$\text{(MRE)} \quad \text{If} \quad a \leq b \quad \text{then} \quad c\backslash a \leq c\backslash b, \quad a/c \leq b/c,$$
$$b\backslash c \leq a\backslash c, \quad c/b \leq c/$$

for all $a, b, c \in M$.

A *model* is a pair $(\mathcal{M}, \mu)$ such that $\mathcal{M}$ is a residuated groupoid with identity and μ is an assignment of elements of M for atoms. One extends μ for all formulas :

$$\mu(\mathbf{1}) = 1, \quad \mu(A \bullet B) = \mu(A) \cdot \mu(B),$$
$$\mu(A\backslash B) = \mu(A)\backslash\mu(B), \quad \mu(A/B) = \mu(A)/\mu(B).$$

and formula structure:

$$\mu(\Lambda) = \mu(\mathbf{1}) = 1, \quad \mu(X \circ Y) = \mu(X) \cdot \mu(Y).$$

A sequent $X \rightarrow A$ is said to be true in model $(\mathcal{M}, \mu)$ if $\mu(X) \leq \mu(A)$. In particular a sequent $\Lambda \rightarrow A$ is said to be true in model $(\mathcal{M}, \mu)$ if $1 \leq \mu(A)$.

One can prove the following property for formula structures:

$$\text{(MON}-\text{STR)} \quad \text{If} \quad \mu(Y) \leq \mu(Z) \quad \text{then} \quad \mu(X[Y]) \leq \mu(X[Z]).$$

5.2 NL1 with assumptions

Let Γ be a set of sequents of the form $A \rightarrow B$, where $A, B \in$ Tp1. By NL1(Γ) we denote the calculus NL1 with additional set Γ of assumptions. NL1 is strongly complete with respect to the residuated groupoids with identity, i.e. all sequents provable in NL1(Γ) are precisely those which are true in all models $(\mathcal{M}, \mu)$ in which all sequents from Γ are true. Soundness is easily proved by induction on derivation in NL1(Γ). Completeness follows from the fact that the Lindenbaum algebra of NL1 is a residuated groupoid with identity.

In general, the calculus NL1(Γ) has not the standard sub-formula property, since (CUT) is legal rule in this system. Thus we take into consideration the sub-formula property in some extended form.

Let T be a set of formulas closed under sub-formulas and such that all formulas appearing in Γ belong to T. By a T-sequent we mean a sequent $X \to A$ such that A and all formulas appearing in X belong to T. Now, we can reformulate the sub-formula property as follows:

> Every T-sequent provable in a system S has a proof in S such that all sequents appearing in this proof are T-sequents.

To prove the sub-formula property for NL1(Γ) we will use special models, namely residuated groupoids with identity of cones over given pre-ordered groupoids with identity.

Let $(M, \leq, \cdot)$ be a pre-ordered groupoid, that means, it is a groupoid with a pre-ordering (i.e. a reflexive and transitive relation), satisfying (MON).

A set $P \subseteq M$ is called a *cone* on M if $a \leq b$ and $b \in P$ entails $a \in P$. Let $C(M)$ denotes the set of cones on M.

The operations $\cdot, \backslash, /$ on $C(M)$ are defined as follows:

$$\text{(M1)} \quad I = \{a \in M : a \leq 1\}$$

$$\text{(M2)} \quad P_1 P_2 = \{c \in M : (\exists a \in P_1, b \in P_2)\ c \leq ab\}$$

$$\text{(M3)} \quad P_1 \backslash P_2 = \{c \in M : (\forall a \in P_1)\ ac \in P_2\}$$

$$\text{(M4)} \quad P_1 / P_2 = \{c \in M : (\forall b \in P_2)\ cb \in P_1\}.$$

A structure $(C(M), \subseteq, \cdot, \backslash, /, I)$ is a residuated groupoid with identity. It is called the residuated groupoid with identity of cones over the given pre-ordered groupoid with identity.

Let M be the set of all formula structures all of whose atomic substructures belong to T and $\Lambda \in M$. If a sequent $X \to A$ has a proof in NL1(Γ) consisting of T-sequents only, we write: $X \to_T A$.

First, we define on M a relation $\leq_b$. $X \leq_b Y$ denotes X directly reduces to Y. The definition of this relation is as follows:

$$Y[Z] \leq_b Y[\Lambda] \quad \text{if} \quad Z \to_T \mathbf{1},$$

$$Y[Z] \leq_b Y[A] \quad \text{if} \quad Z \to_T A,$$

$$Y[A \bullet B] \leq_b Y[A \circ B] \quad \text{if} \quad A \bullet B \in T.$$

A pre-ordering $\leq$ on M is defined as a reflexive and transitive closure of the relation $\leq_b$. Then $X \leq Y$ iff there exist $Y_0, \ldots, Y_n, n \geq 0$ such that $X = Y_0, Y = Y_n$ and $Y_{i-1} \leq_b Y_i$, for each $i = 1, \ldots, n$.

Clearly, $(M, \leq, \circ, \Lambda)$ is a pre-ordered groupoid with identity Λ fulfilling (MON).

Next, we take into consideration the residuated groupoid of cones with identity $C(M) = (C(M), \subseteq, \cdot, \backslash, /, I)$ over $(M, \leq, \circ, \Lambda)$ defined above. An assignment μ on $C(M)$ is defined by setting:

$$\mu(p) = \{X \in M : X \to_T p\},$$

for all atoms p. One can easily prove that

$$\mu(A) = \{X \in M : X \rightarrow_T A\},$$

for all $A \in T$.

Fact 1 *Every sequent provable in* NL1(Γ) *is true in* $(C(M), \mu)$.

Proof. It suffice to show, that each axiom from Γ is true in $(C(M), \mu)$. Assume that $A \rightarrow B$ belongs to Γ. It yields $A \rightarrow_T B$. We need to show that $\mu(A) \subseteq \mu(B)$. Let $X \in \mu(A)$. Then, $X \rightarrow_T A$. By (CUT), we get $X \rightarrow_T B$, which yields $X \in \mu(B)$. □

Lemma 2 *The system* NL1(Γ) *has the extended sub-formula property.*

Proof. Let $X \rightarrow A$ be a T-sequent provable in NL1(Γ). By fact 1 it is true in the model (C, μ), i.e. $\mu(X) \subseteq \mu(A)$. Since $X \in \mu(X)$, we have $X \in \mu(A)$. But it means $X \rightarrow_T A$. □

A sequent is said to be *basic* if it is a T-sequent of the form $\Lambda \rightarrow A$, $A \rightarrow B$, $A \circ B \rightarrow C$. Let Γ be finite, and let T be a finite set of formulas, closed under sub-formulas and such that T contains all formulas appearing in Γ. For such T we shall describe an effective procedure which produces all basic sequents derivable in NL1(Γ).

Let S_0 consist of all T-sequent of the form (Id), all sequents from Γ, $\Lambda \rightarrow \mathbf{1}$ and all T-sequents of the form:

$$\mathbf{1} \circ A \rightarrow A,\ A \circ \mathbf{1} \rightarrow A,\ A \circ B \rightarrow A \bullet B,$$
$$A \circ (A \backslash B) \rightarrow B,\ (A/B) \circ B \rightarrow A.$$

Assume S_n has already been defined. S_{n+1} is S_n enriched with sequents resulting from the following rules:

(S1) if $(A \circ B \rightarrow C) \in S_n$ and $(A \bullet B) \in T$, then $(A \bullet B \rightarrow C) \in S_{n+1}$,
(S2) if $(A \circ X \rightarrow C) \in S_n$ and $(A \backslash C) \in T$, then $(X \rightarrow A \backslash C) \in S_{n+1}$,
(S3) if $(X \circ B \rightarrow C) \in S_n$ and $(C/B) \in T$, then $(X \rightarrow C/B) \in S_{n+1}$,
(S4) if $(\Lambda \rightarrow A) \in S_n$ and $(A \circ X \rightarrow C) \in S_n$, then $(X \rightarrow C) \in S_{n+1}$,
(S5) if $(\Lambda \rightarrow A) \in S_n$ and $(X \circ A \rightarrow C) \in S_n$, then $(X \rightarrow C) \in S_{n+1}$,
(S6) if $(A \rightarrow B) \in S_n$ and $(B \circ X \rightarrow C) \in S_n$, then $(A \circ X \rightarrow C) \in S_{n+1}$,
(S7) if $(A \rightarrow B) \in S_n$ and $(X \circ B \rightarrow C) \in S_n$, then $(X \circ A \rightarrow C) \in S_{n+1}$,
(S8) if $(A \circ B \rightarrow C) \in S_n$ and $(C \rightarrow D) \in S_n$, then $(A \circ B \rightarrow D) \in S_{n+1}$.

Clearly, $S_n \subseteq S_{n+1}$ for all $n \geq 0$. We define S^T as the join of this chain. S^T is a set of basic sequents, hence it must be finite. It yields $S^T = S_{k+1}$, for the least k such that $S_k = S_{k+1}$, and this k is not greater then the number of basic sequents.

Fact 3 *The set* S^T *can be constructed in polynomial time.*

Proof. Let n be the cardinality of T. There are n, n^2 and n^3 basic sequents of the form $\Lambda \to A$, $A \to B$ and $A \circ B \to C$, respectively. Hence, we have $m = n^3 + n^2 + n$ basic sequents. The set S_0 can be constructed in time $0(n^2)$. To get S_{i+1} from S_i we must close S_i under the rules (S1)-(S8) which can be done in at most m^3 steps for each rule. For example, to close S_i under (S1) we must check if $(A \circ B \to C) \in S_i$ and $(A \bullet B) \in T$ which needs at most m and n steps, respectively. The sequent $A \bullet B \to C$ is added to S_{i+1} only if it doesn't belong to this set. To check this fact the next m steps are needed. The least k such that $S^T = S_k$ is at most m. Then finely, we can construct S^T from T in time $0(m^4) = 0(n^{12})$. □

By $S(T)$ we denote the system whose axioms are all sequents from S^T and whose only inference rule is (CUT). Then, every proof in $S(T)$ consist of T-sequents only.

If as premises of (CUT) in the proof in $S(T)$ of some sequent $X \to A$ only sequents without empty antecedents are used, then the length of all sequents in this proof is not greater than the length of $X \to A$. But it doesn't hold if we allow in (CUT) the premises of the form $\Lambda \to A$. Therefore we introduce another system $S(T)^-$ whose axioms are all sequents from S^T and whose only inference rule is (CUT) with premises without empty antecedents, and show the following lemma.

Lemma 4 *For any sequent* $X \to A$, $S(T) \vdash X \to A$ *iff* $S(T)^- \vdash X \to A$.

Proof. The 'if' direction is evident. To prove the 'only if' direction we show that $S(T)^-$ is closed under (CUT), i.e.

(*) If $S(T)^- \vdash X \to B$ and $S(T)^- \vdash Y[B] \to A$, then $S(T)^- \vdash Y[X] \to A$.

Assume $S(T)^- \vdash X \to B$ and $S(T)^- \vdash Y[B] \to A$.

If $X \neq \Lambda$, then $S(T)^- \vdash Y[X] \to A$ by definition of $S(T)^-$.

If $X = \Lambda$, then the sequent $X \to B$ is of the form $\Lambda \to B$ and $S(T)^- \vdash \Lambda \to B$, which means that $\Lambda \to B$ is an axiom of $S(T)^-$. To prove (*) we proceed by induction on derivation of the second premise: $Y[B] \to A$.

If $Y[B] \to A$ is an axiom of $S(T)^-$, then $(Y[B] \to A) \in S^T$. S^T is closed under (CUT). Hence, $(Y[\Lambda] \to A) \in S^T$ which yields $S(T)^- \vdash Y[\Lambda] \to A$.

If $Y[B] \to A$ is a conclusion of (CUT) from premises without empty antecedents, then $Y[B] = Z[Y']$ and for some $C \in T$, $S(T)^- \vdash Y' \to C$ and $S(T)^- \vdash Z[C] \to A$. We consider the following cases.

I. B is contained in Y'. Then $Y' = Y'[B]$.

(1) $Y'[B] \neq B$. By the induction hypothesis, (*) holds for $\Lambda \to B$ and $Y'[B] \to C$, so $S(T)^- \vdash Y'[\Lambda] \to C$. Since $Y'[B] \neq B$, we have $Y'[\Lambda] \neq \Lambda$. Using (CUT), we get $S(T)^- \vdash Z[Y'[\Lambda]] \to A$, which means $S(T)^- \vdash Y[\Lambda] \to A$.

(2) $Y'[B] = B$. By the induction hypothesis, (*) holds for $\Lambda \to B$ and

$B \to C$, so $S(T)^- \vdash \Lambda \to C$. Using inductive hypothesis to $\Lambda \to C$ and $Z[C] \to A$, we get $S(T)^- \vdash Z[\Lambda] \to A$, which means $S(T)^- \vdash Y[\Lambda] \to A$.

II. B and Y' do not overlap. Then B is contained in Z and does not overlap C in Z. We write $Z[C] = Z[B, C]$. From the assumption we have $Y' \neq \Lambda$. By induction hypothesis, (*) holds for $\Lambda \to B$ and $Z[B, C] \to A$, so $S(T)^- \vdash Z[\Lambda, C] \to A$. By (CUT), $S(T)^- \vdash Z[\Lambda, Y'] \to A$, which means $S(T)^- \vdash Y[\Lambda] \to A$.

□

Corollary 5 *Every basic sequents provable in $S(T)$ belongs to S^T.*

Proof. We proceed by induction on proofs in $S(T)$. Assume $X \to A$ is a basic sequent derivable in $S(T)$. If $X \to A$ is an axiom of $S(T)$, then $(X \to A) \in S^T$. If $X \to A$ is a conclusion of (CUT), we consider three cases.

(1) $X = \Lambda$. By lemma 4, $\Lambda \to A$ has a proof in $S(T)^-$. Hence $\Lambda \to A$ is an axiom, which means $(\Lambda \to A) \in S^T$.

(2) $X = B$. By lemma 4, there exists a proof such that $B \to A$ is a conclusion from premises $B \to C$ and $C \to A$, where $C \neq \Lambda$. Since proofs in S(T) consist with T-sequents only, $B \to C$ and $C \to A$ are basic sequents. By induction hypothesis, $(B \to C) \in S^T$ and $(C \to A) \in S^T$. S^T is closed under (CUT), so $(B \to A) \in S^T$.

(3) $X = B \circ C$. By lemma 4, there exists a proof such that $B \circ C \to A$ is a conclusion from premises without empty premises. Hence, they are of the form: $(B \circ C \to D\, ,\, D \to A)$ or $(B \to D\, ,\, D \circ C \to A)$ or $(C \to D\, ,\, B \circ D \to A)$. By the same argument as in (2), in each case, we get $(B \circ C \to A) \in S^T$.

□

Now, we can state an interpolation lemma for $S(T)$.

Lemma 6 *If $S(T) \vdash X[Y] \to A$, then there exists $D \in T$ such that $S(T) \vdash Y \to D$ and $S(T) \vdash X[D] \to A$.*

Proof. We proceed by induction on proofs in $S(T)$.

I. Assume $X[Y] \to A$ is an axiom of $S(T)$. We consider the following cases.

(1) $X[Y] = Y$. Then $Y = X$ (observe, that this case includes sub case $X = \Lambda$). We set $D = A$. We have $S(T) \vdash X \to A$ from assumption and $S(T) \vdash A \to A$, since $(A \to A) \in S^T$.

(2) $X[Y] = B$, $Y = \Lambda$. Then $X[Y] = X[\Lambda] = B = B \circ \Lambda$ or $X[Y] = \Lambda \circ B$ and $D = \mathbf{1}$. We have $S(T) \vdash \Lambda \to \mathbf{1}$ and $S(T) \vdash B \to A$. $(B \circ \mathbf{1} \to B) \in S^T$, so $S(T) \vdash B \circ \mathbf{1} \to B$. Using (CUT) we get $S(T) \vdash X[\mathbf{1}] \to A$. For $X[Y] = \Lambda \circ B$ the argument is dual.

(3) $X[Y] = B \circ C$, $Y \neq \Lambda$. Then $Y = B$ or $Y = C$, hence $D = B$ or $D = C$, respectively.

(4) $X[Y] = B \circ C$, $Y = \Lambda$. Then $X[\Lambda]$ has one of the form: $\Lambda \circ (B \circ C)$, $(B \circ C) \circ \Lambda$, $(\Lambda \circ B) \circ C$, $(B \circ \Lambda) \circ C$, $B \circ (\Lambda \circ C)$, $B \circ (C \circ \Lambda)$. In all these cases we set $D = \mathbf{1}$. For example, if $X[\Lambda] = \Lambda \circ (B \circ C)$, we have $S(T) \vdash \Lambda \to \mathbf{1}$ and using (CUT) to $S(T) \vdash B \circ C \to A$ and $S(T) \vdash \mathbf{1} \circ A \to A$, we get $S(T) \vdash \mathbf{1} \circ (B \circ C) \to A$.

II. Assume $X[Y] \to A$ is a conclusion of (CUT). Then $X[Y] = Z[Y']$ and for some $B \in T$: $S(T) \vdash Y' \to B$ and $S(T) \vdash Z[B] \to A$.

In this part the proof is analogous to the proof of lemma 2 in Buszkowski (2005). The following cases are considered.

(1) Y is contained in Y'. Then $Y' = Y'[Y]$. By the induction hypothesis, there exists $D \in T$ such that $S(T) \vdash Y \to D$ and $S(T) \vdash Y'[D] \to B$. Using (CUT) with the premises $Z[B] \to A$ and $Y'[D] \to B$ we get $S(T) \vdash Z[Y'[D]] \to A$, which means $S(T) \vdash X[D] \to A$.

(2) Y' is contained in Y. Then $X[Y] = X[Y[Y']] = Z[Y']$ and $Z[B] = X[Y[B]]$. By the induction hypothesis, there exists $D \in T$ such that $S(T) \vdash Y[B] \to D$ and $S(T) \vdash X[D] \to A$. Using (CUT) with the premises $Y' \to B$ and $Y[B] \to D$ we get $S(T) \vdash Y[Y']] \to D$.

(3) Y and Y' do not overlap. Then Y is contained in Z and does not overlap B in Z. We write $Z[B] = Z[B, Y]$. By the induction hypothesis, there exists $D \in T$ such that $S(T) \vdash Y \to D$ and $S(T) \vdash Z[B, D] \to A$. Using (CUT) with the premises $Y' \to B$ and $Z[B, D] \to B$ we get $S(T) \vdash Z[Y', D] \to A$, which means $S(T) \vdash X[D] \to A$.

□

Lemma 7 *For any T-sequent $X \to A$, $X \to_T A$ iff $S(T) \vdash X \to A$.*

Proof. Recall, that $X \to_T A$ means that the sequent $X \to A$ has the proof in NL1(Γ) consisting with T-sequents only.

To prove 'if' direction observe that $X \to_T A$, for all sequents $X \to A$ in S^T.

The T-sequents which are axioms of NL1(Γ) belong to S_0. Thus, to prove the 'only if' direction it suffices to show that all inference rules of NL1(Γ), restricted to T-sequents, are admissible in $S(T)$. For example, let us consider ($\mathbf{1}$L). Assume $X[\Lambda] \to A$. By lemma 6, there exist $D \in T$ such that $S(T) \vdash \Lambda \to D$ and $S(T) \vdash X[D] \to A$. Since $(D \circ \mathbf{1} \to D) \in S^T$, then $S(T) \vdash D \circ \mathbf{1} \to D$. By two applications of (CUT), we get $S(T) \vdash X[\Lambda \circ \mathbf{1}] \to A$, which means $S(T) \vdash X[\mathbf{1}] \to A$. □

Theorem 8 *If Γ is finite, then* NL1(Γ) *is decidable in polynomial time.*

Proof. Let Γ be a finite set of sequents of the form $B \to C$ and let $X \to A$ be a sequent. Let n be the number of logical constants and atoms in $X \to A$

and Γ. As T we choose the set of all sub-formulas of formulas appearing in $X \to A$ and formulas appearing in Γ. Since the number of sub-formulas of any formula B is equal to the number of logical constants and atoms in B, T has n elements and we can construct it in time $0(n^2)$. By lemma 2, $\mathrm{NL1}(\Gamma) \vdash X \to A$ iff $X \to_T A$. By lemma 7, $X \to_T A$ iff $S(T) \vdash X \to A$. Proofs in $S(T)$ are actually derivation trees of a context-free grammar whose production rules are the reversed sequents from S^T. Checking derivability in context-free grammars is P-TIME decidable. For example, by known CYK algorithm, it can be done in time not exceed $k \cdot n^3$, where k is the size of S^T. By the proof of fact 3, the size of S^T is at most $0(n^3)$ and S^T can be constructed in $0(n^{12})$. Hence, the total time is $0(n^{12})$, i.e. NL1(Γ) is P-TIME decidable. □

By theorem 8, we have immediately that languages generated by the categorial grammar based on the system NL1(Γ) are context-free. In Buszkowski (2005) the analogous result was established for NL(Γ), NL(Γ) with permutation rule and Generalized Lambek Calculus (GLC(Γ)). The context-freeness of the languages generated by Non-associative Lambek Calculus were studied by Buszkowski (1986), Kandulski (1988) and Jäger (2004). Bulińska (2005) obtained the weak equivalence of context-free grammars and grammars based on the associative Lambek calculus with finite set of simple non-logical axioms of the form $p \to q$, where p, q are primitive types.

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