9

Pregroups with modalities

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Abstract

In this paper we concentrate mainly on the notion of β-pregroups, which are pregroups (first introduced by Lambek Lambek (1999) in 1999) enriched with modality operators. β-pregroups were first proposed by Fadda Fadda (2002) in 2001. The motivation to introduce them was to (locally) limit the associativity in the calculus considered. In this paper we present this new calculus in the form of a rewriting system, and prove the very important feature of this system: that in a given derivation the non-expanding rules must always precede non-contracting ones in order for the derivation to be minimal (normalization theorem). We also propose a sequent system for this calculus and prove the cut elimination theorem for it.

Keywords Pregroup, β-pregroup, normalization theorem, cut elimination

9.1 Introduction

Definition 2 A pregroup is a structure $(G, \leq, \cdot, l, r, 1)$ such that $(G, \leq, \cdot, 1)$ is a partially ordered monoid, and l, r are unary operations on G, fulfilling the following conditions:

$$a^l a \leq 1 \leq aa^l \textit{ and } aa^r \leq 1 \leq a^r a \tag{9.1}$$

for all $a \in G$. Element a^l (a^r respectively) is called the left (right) adjoint of a.

The notion of a pregroup, introduced by Lambek Lambek (1999), is connected to the notion of a residuated monoid, known from the theory of partially ordered algebraic systems.

Theorem 18 (Lambek (1999)) *In each pregroup the following equalities and inequalities are valid:*

$$1^l = 1^r = 1, \quad a^{lr} = a = a^{rl}, \tag{9.2}$$

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$$(ab)^l = b^l a^l, \quad (ab)^r = b^r a^r, \tag{9.3}$$

$$a \leq b \quad \textit{iff} \quad b^l \leq a^l \quad \textit{iff} \quad b^r \leq a^r. \tag{9.4}$$

For any arbitrary element a of a pregroup we define an element $a^{(n)}$, for $n \in Z$, in a following way: $a^0 = a$, $a^{(n+1)} = (a^{(n)})^r$, $a^{(n-1)} = (a^{(n)})^l$. As a consequence of (2) and (9.4) we obtain:

$$a^{(n)} a^{(n+1)} \leq 1 \leq a^{(n+1)} a^{(n)} \tag{9.5}$$

$$if\ a \leq b\ then\ a^{(2n)} \leq b^{(2n)}\ and\ b^{(2n+1)} \leq a^{(2n+1)} \tag{9.6}$$

for all $n \in Z$.

Let $(P, \leq)$ be a poset. Elements of the set P are treated as constants. *Terms* are expressions of the form $p^{(n)}$, for $p \in P$, $n \in Z$; $p^{(0)}$ is equal p. *Types* are finite strings of terms, denoted by X, Y, Z, V, U etc. The basic rewriting rules are as follows:

- (CON) - contraction:
  $X, p^{(n)}, p^{(n+1)}, Y \rightarrow X, Y$;
- (EXP) - expansion:
  $X, Y \rightarrow X, p^{(n+1)}, p^{(n)}, Y$;
- (IND) - induced step:
  $X, p^{(2n)}, Y \rightarrow X, q^{(2n)}, Y,$
  $X, q^{(2n+1)}, Y \rightarrow X, p^{(2n+1)}, Y,$ for $p \leq q$ w $(P, \leq)$.

Furthermore, we consider derivations $X \Rightarrow Y$ in $F(P)$ (free pregroup generated by $(P, \leq)$). Following Lambek (2001), we distinguish two special cases:

- $(GCON)$ - generalized contraction:
  $X, p^{(2n)}, q^{(2n+1)}, Y \rightarrow X, Y$;
  $X, q^{(2n-1)}, p^{(2n)}, Y \rightarrow X, Y$; where $p \leq q$ in $(P, \leq)$.
- $(GEXP)$ - generalized expansion:
  $X, Y \rightarrow X, p^{(2n+1)}, q^{(2n)}, Y$;
  $X, Y \rightarrow X, q^{(2n)}, p^{(2n-1)}, Y$; where $p \leq q$ in $(P, \leq)$.

The relation $\Rightarrow$ is a reflexive and transitive closure of the relation $\rightarrow$.

Theorem 19 (Lambek switching lemma, Lambek (1999)) *If $X \Rightarrow Y$ is in $F(P)$, then there exist types U, V such that we can go from type X to U ($X \Rightarrow U$) using only generalized contractions, from type U to V ($U \Rightarrow V$) using only induced steps, and from type V to Y ($V \Rightarrow Y$) using only generalized expansions.*

From the above mentioned lemma we obtain:

Corollary 20 (Buszkowski (2003)) *If $X \Rightarrow Y$ in $F(P)$, and Y is a simple type or an empty string, then X can be transformed into Y only by means of* (CON)

and (IND)*. If* $X \Rightarrow Y$ *in* $F(P)$*, and* X *is a simple type or an empty string, then* X *can be transformed into* Y *only by means of* (EXP) *and* (IND)*.*

9.2 Pregroups with modalities

In this section we generalize some definitions and results concerning pregroups introduced in Lambek (1999). The definition of a pregroup with β-operator was proposed by Fadda (2002). The motivation to introduce modality operators stems from the fact there was a need to (locally) limit associativity in the calculus considered.

Definition 3 A pregroup with β-operator is a pregroup G enriched additionally with a monotone mapping $\beta : G \to G$.

Definition 4 A β-pregroup is a pregroup with β-operator such that β-operator has the right adjoint $\hat{\beta}$ ($\hat{\beta}$-operator), i.e., there exists a monotone mapping $\hat{\beta} : P \to P$ with the property that for all a and b in P, $\beta(a) \leq b$ if and only if $a \leq \hat{\beta}(b)$.

It is easy to show that $\hat{\beta}$-operators, if they exist, are uniquely defined and connected to β-operators with the following rules of expansion and contraction, for all $a \in P$.

$$a \leq \hat{\beta}(\beta(a)) \quad \text{and} \quad \beta(\hat{\beta}(a)) \leq a. \tag{9.7}$$

The basic rewriting rules are as follows:

1. **Contracting rules**
   - (CON) - contraction:
     $X, p^{(n)}, p^{(n+1)}, Y \to X, Y;$
   - $(B - CON)$ - B-contraction:
     $X, [B(Y)]^{(n)}, [B(Y)]^{(n+1)}, Z \to X, Z;$ where $B \in \{\beta, \hat{\beta}\}$.
   - $(\beta - CON)$ - β-contraction:
     $X, [\beta(\hat{\beta}(Y))]^{(2n)}, Z \to X, Y^{(2n)}, Z;$
     $X, [\hat{\beta}(\beta(Y))]^{(2n+1)}, Z \to X, Y^{(2n+1)}, Z;$
   - $(B - IND_c)$ - B_c induced step:
     $X, [B(Y_1)]^{(2n)}, Z \to X, [B(Y_2)]^{(2n)}, Z;$
     where $B \in \{\beta, \hat{\beta}\}$, and $Y_1 \to Y_2$ is a contracting rule.
     $X, [B(Y_2)]^{(2n+1)}, Z \to X, [B(Y_1)]^{(2n+1)}, Z;$
     where $B \in \{\beta, \hat{\beta}\}$, a $Y_1 \to Y_2$ is an expanding rule.
2. **Expanding rules**
   - (EXP) - expansion:
     $X, Y \to X, p^{(n+1)}, p^{(n)}, Y;$
   - $(B - EXP)$ - B-expansion:
     $X, Z \to X, [B(Y)]^{(n+1)}, [B(Y)]^{(n)}, Z;$ where $B \in \{\beta, \hat{\beta}\}$.

- $(\beta - EXP)$ - β - expansion:
  $X, Y^{(2n)}, Z \rightarrow X, [\hat{\beta}(\beta(Y))]^{(2n)}, Z;$
  $X, Y^{(2n+1)}, Z \rightarrow X, [\beta(\hat{\beta}(Y))]^{(2n+1)}, Z.$
- $(B - IND_e)$ - B_e induced step:
  $X, [B(Y_1)]^{(2n)}, Z \rightarrow X, [B(Y_2)]^{(2n)}, Z;$
  where $B \in \{\beta, \hat{\beta}\}$, a $Y_1 \rightarrow Y_2$ is an expanding rule.
  $X, [B(Y_2)]^{(2n+1)}, Z \rightarrow X, [B(Y_1)]^{(2n+1)}, Z;$
  where $B \in \{\beta, \hat{\beta}\}$, a $Y_1 \rightarrow Y_2$ is a contracting rule.

3. **P-rules (neither expanding nor contracting)**

- (IND) - induced step:
  $X, p^{(2n)}, Y \rightarrow X, q^{(2n)}, Y,$
  $X, q^{(2n+1)}, Y \rightarrow X, p^{(2n+1)}, Y,$ for $p \leq q$ w $(P, \leq)$.
- $(B - IND_p)$ - B_p induced step:
  $X, [B(Y_1)]^{(2n)}, Z \rightarrow X, [B(Y_2)]^{(2n)}, Z;$
  where $B \in \{\beta, \hat{\beta}\}$, and $Y_1 \rightarrow Y_2$ is a P-rule.
  $X, [B(Y_2)]^{(2n+1)}, Z \rightarrow X, [B(Y_1)]^{(2n+1)}, Z;$
  where $B \in \{\beta, \hat{\beta}\}$, and $Y_1 \rightarrow Y_2$ is a P-rule.

In the above mentioned rules we assume that p, q are elements of P, whereas X, Y, Z, Y_1, Y_2 are elements of P'. The relation $\Rightarrow$ is a reflexive and transitive closure of the relation $\rightarrow$.

Fadda (2002) gives some examples illustrating the usage of β - pregroups for natural language. Among others, he shows that assigning a type $[\beta(X)]^r X [\beta(X)]^l$ to the conjunction *and* (where X is an arbitrary type), will let us see the structure of a sentence more clearly.

Consider the sentence: *John and Mary left.* Applying the calculus of pregroups without modalities we can show that the string of types assigned to given words can be reduced to the type of a sentence. However, the order of consecutive contraction is important here (np means a noun phrase):

(*)	**John**	**and**	**Mary**	**left.**		
	np	$np^r\ np\ np^l$	np	$np^r s$	$\rightarrow$	
		$np\ np^l$	np	$np^r s$	$\rightarrow$	
		np		$np^r s$	$\rightarrow$	s
(**)	**John**	**and**	**Mary**	**left.**		
	np	$np^r\ np\ np^l$	np	$np^r s$	$\rightarrow$	
		$np\ np^l$	np	$np^r s$	$\rightarrow$	
		$np\ np^l$		s	$\nrightarrow$	s

In the second case (**) we do not get a type s. Applying the calculus of β-pregroups, we could handle the above mentioned sentence in the following way:

(**)	**John**	**and**	**Mary**	**left.**		
	$\beta(np)$	$[\beta(np)]^r np\ [\beta(np)]^l$	$\beta(np)$	$np^r s$	$\rightarrow$	s

In that case the structure of types ’induces’ the order of contractions.

Normalization theorem for β - pregroups

Further we consider derivations of a type $X \Rightarrow Y$.

Definition 5 A derivation is called non-expanding, if there are no expanding rules present.

Definition 6 A derivation is called non-contracting, if there are no contracting rules present.

Definition 7 Composition of derivations $d_1(X \Rightarrow U)$ and $d_2(U \Rightarrow Y)$ is a derivation Y from X, which transforms first X into U according to d_1, and then U into Y according to d_2.

Definition 8 A derivation $d(X \Rightarrow Y)$ is called normal, if it is a composition of some non-expanding derivation $d_1(X \Rightarrow U)$ and some non-contracting derivation $d_2(U \Rightarrow Y)$.

On elements of P' we introduce a measure in the following way:

$\mu(\varepsilon) = 0,$
$\mu(p^{(n)}) = 1,$
$\mu(B(Y)) = \mu(Y) + 1, \quad \text{for } B \in \{\beta, \hat{\beta}\}$
$\mu(Y_1, ..., Y_k) = \mu(Y_1) + ... + \mu(Y_k).$

A measure on the rewriting rules is defined as follows:

$\mu(CON) = 2,$
$\mu(EXP) = 2,$
$\mu(\beta - CON) = 2,$
$\mu(\beta - EXP) = 2,$
$\mu(B - CON) = 2 + 2\mu(Y),$
$\mu(B - EXP) = 2 + 2\mu(Y),$
$\mu(IND) = 1,$
$\mu(B_c - IND) = 1 + \mu(d(Y_1 \to Y_2)),$
$\mu(B_e - IND) = 1 + \mu(d(Y_1 \to Y_2)),$
$\mu(B_p - IND) = 1 + \mu(d(Y_1 \to Y_2)),$
$\mu(d(X_0 \Rightarrow X_k)) = \mu(d(X_0 \to X_1)) + ... + \mu(d(X_{k-1} \to X_k)),$
where $X_0 \Rightarrow X_k$ means $X_0 \to X_1 \to ... \to X_k$.

Definition 9 A derivation $d(X \Rightarrow Y)$ is called minimal, if it has the least possible measure of all derivations Y from X, and the least possible complexity (which is understood as a sum of measures of all rules used in the derivation).

Definition 10 The position of a given rule in the derivation $X_0 \to X_1 \to ... \to X_n$ is number i, such that $X_{i-1} \to X_i$ is the occurrence of this rule in the derivation.

Definition 11 A degree of non-normal derivation $d(X \Rightarrow Y)$ is the minimal position of a contracting rule which occurs (not necessarily directly) after an expanding rule. A degree of normal derivation is number 0.

Theorem 21 (Normalization theorem for β-pregroups) *Every minimal derivation is normal.*

Proof. Let $X_0 \to X_1 \to ... \to X_n$ be a minimal derivation. Let i be a degree of this derivation. We will show that $i = 0$, and as a consequence our derivation is normal. Assume that $i > 0$. Of course $1 < i \leq n$ from the definition of a degree. Let j be the greatest number less than i, such that $X_{j-1} \to X_j$ is the occurrence of an expanding rule.
Let R_1 denote the rule used on the position j, and R_2 the rule used in the position i. The following cases are to be considered:

1.1. $R_1 = (EXP)$ $R_2 = (CON)$,
1.2. $R_1 = (EXP)$ $R_2 = (B - CON)$,
1.3. $R_1 = (EXP)$ $R_2 = (\beta - CON)$,
1.4. $R_1 = (EXP)$ $R_2 = (B - IND_c)$,
2.1. $R_1 = (B - EXP)$ $R_2 = (CON)$,
2.2. $R_1 = (B - EXP)$ $R_2 = (B - CON)$,
2.3. $R_1 = (B - EXP)$ $R_2 = (\beta - CON)$,
2.4. $R_1 = (B - EXP)$ $R_2 = (B - IND_c)$,
3.1. $R_1 = (\beta - EXP)$ $R_2 = (CON)$,
3.2. $R_1 = (\beta - EXP)$ $R_2 = (B - CON)$,
3.3. $R_1 = (\beta - EXP)$ $R_2 = (\beta - CON)$,
3.4. $R_1 = (\beta - EXP)$ $R_2 = (B - IND_c)$,
4.1. $R_1 = (B - IND_e)$ $R_2 = (CON)$,
4.2. $R_1 = (B - IND_e)$ $R_2 = (B - CON)$,
4.3. $R_1 = (B - IND_e)$ $R_2 = (\beta - CON)$,
4.4. $R_1 = (B - IND_e)$ $R_2 = (B - IND_c)$,

In the proof of this theorem the above mentioned cases are considered. In all cases we assume that the rule R_1 occurs on the position j, and the rule R_2 on the position i. All steps $X_j \to X_{j+1} \to ... \to X_{i-1}$ consist of application of non-expanding and non-contracting rules. These must be of the form of either (IND) or $(B_p - IND)$. None of this steps cannot be independent from $X_{i-1} \to X_i$, as otherwise we could do the last of independent steps after R_2, getting the derivation with the same measure but the lower degree. We can also assume that none of this steps is not independent from $X_{j-1} \to X_j$; otherwise it would transform our derivation performing the first step before R_1, increasing the number j, and changing neither i nor $\mu(d(X \Rightarrow Y))$.

If the rules R_1 and R_2 are adjacent (without intermediate P-rules), we change the order in case they are independent from each other (getting the derivation of smaller complexity); in case they are dependent from each other we show that this part of derivation can be transformed using rules of smaller complexity - thus showing that the initial derivation was not normal.

Considering the sixteen cases mentioned above, we show that non-expanding rules must always precede non-contracting ones. Otherwise our derivation would not be minimal, which would be a contradiction to our assumption. Thus every minimal derivation must be normal.

As the proof is long and technical, we show as an example only one of above mentioned sixteen cases:

Case 1.1. $R_1 = (EXP)$ $R_2 = (CON)$,
$X_{j-1} \to X_j$ is of the form $S, T \to S, p^{(n+1)}, p^{(n)}, T$; $X_{i-1} \to X_i$ is of the form $U, q^{(n)}, q^{(n+1)}, V \to U, V$. The derivation $X_{j-1} \to X_j \to ... \to X_{i-1} \to X_i$ could be as follows:
$S, p_0^{(2n)}, T \to S, p_0^{(2n)}, p_k^{(2n+1)}, p_k^{(2n)}, T \to S, p_0^{(2n)}, p_{k-1}^{(2n+1)}, p_k^{(2n)}, T \to ...$
$\to S, p_0^{(2n)}, p_0^{(2n+1)}, p_k^{(2n)}, T \to S, p_k^{(2n)}, T$, (assuming $p_0 \leq p_1 \leq ... \leq p_k$), its measure is $\mu(d(X_{j-1} \Rightarrow X_i)) = 2 + k + 2 = k + 4$.

The above mentioned derivation can be changed by the derivation:
$S, p_0^{(2n)}, T \to S, p_1^{(2n)}, T \to ...S, p_{k-1}^{(2n)}, T \to S, p_k^{(2n)}, T$, (assuming $p_0 \leq p_1 \leq ... \leq p_k$). The measure of a new derivation is $\mu(d(X_{j-1} \Rightarrow X_i)) = k$ (k times the rule (IND) was used). We reach a contradiction, as the measure of the second derivation is smaller. We showed that the initial derivation was not normal. □

Corollary 22 *If $X \Rightarrow Y$ in a free β-pregroup, and Y is a simple type or an empty string, then Y can be derived from X only by means of non-expanding rules.*

If $X \Rightarrow Y$ in a free β-pregroup, and X is a simple type or an empty string, then Y can be derived from X only by means of non-contracting rules.

9.3 Axiom system for pregroups with modalities

The rewriting system given in the previous section can also be presented as the calculus of sequents in a Gentzen style. Let $(P, \leq)$ be fixed. Atoms and types are defined as before. *Sequents* are of the form $X \Rightarrow Y$, where X, Y are types. The axiom and inference rules are as follows:

(Id) $X \Rightarrow X$,

(LA) $$\frac{X, Y \Rightarrow Z}{X, p^{(n)}, p^{(n+1)}, Y \Rightarrow Z}$$ (RA) $$\frac{X \Rightarrow Y, Z}{X \Rightarrow Y, p^{(n+1)}, p^{(n)}, Z}$$

(LIND) $$\frac{X, q^{(2n)}, Y \Rightarrow Z}{X, p^{(2n)}, Y \Rightarrow Z} \qquad \frac{X, p^{(2n+1)}, Y \Rightarrow Z}{X, q^{(2n+1)}, Y \Rightarrow Z}$$

(RIND) $$\frac{X \Rightarrow Y, p^{(2n)}, Z}{X \Rightarrow Y, q^{(2n)}, Z} \qquad \frac{X \Rightarrow Y, q^{(2n+1)}, Z}{X \Rightarrow Y, p^{(2n+1)}, Z}$$

In rules (LIND) and (RIND) we assume that $p \leq q$ in P. X, Y, Z are any arbitrary types, p, q are arbitrary elements of P, for $n \in Z$.

(BLA) $\dfrac{X, T \Rightarrow Z}{X, [B(Y)]^{(n)}, [B(Y)]^{(n+1)}, T \Rightarrow Z}$

(BRA) $\dfrac{X \Rightarrow T, Z}{X \Rightarrow T, [B(Y)]^{(n+1)}, [B(Y)]^{(n)}, Z}$

(β LA) $\dfrac{X, Y^{(2n)}, T \Rightarrow Z}{X, [\beta(\hat{\beta}(Y))]^{(2n)}, T \Rightarrow Z}$ (β RA) $\dfrac{X \Rightarrow T, Y^{(2n)}, Z}{X \Rightarrow T, [\hat{\beta}(\beta(Y))]^{(2n)}, Z}$

$\dfrac{X, Y^{(2n+1)}, T \Rightarrow Z}{X, [\hat{\beta}(\beta(Y))]^{(2n+1)}, T \Rightarrow Z}$ $\dfrac{X \Rightarrow T, Y^{(2n+1)}, Z}{X \Rightarrow T, [\beta(\hat{\beta}(Y))]^{(2n+1)}, Z}$

(BLIND) $\dfrac{X, [B(Y_2)]^{(2n)}, Z \Rightarrow T}{X, [B(Y_1)]^{(2n)}, Z \Rightarrow T}$ (BRIND) $\dfrac{X \Rightarrow T, [B(Y_1)]^{(2n)}, Z}{X \Rightarrow T, [B(Y_2)]^{(2n)}, Z}$

$\dfrac{X, [B(Y_1)]^{(2n+1)}, Z \Rightarrow T}{X, [B(Y_2)]^{(2n+1)}, Z \Rightarrow T}$ $\dfrac{X \Rightarrow T, [B(Y_2)]^{(2n+1)}, Z}{X \Rightarrow T, [B(Y_1)]^{(2n+1)}, Z}$

In rules (BLA), (BRA), (BLIND) and (BRIND), $B \in \{\beta, \hat{\beta}\}$. Additionally, in rules (BLIND) we assume that $Y_1 \rightarrow Y_2$ arises as a result of a non-expanding rule in an even case, and a non-contracting rules in an odd case, in a rewriting system from a former section. In rules (BRIND) we assume that $Y_1 \rightarrow Y_2$ arises as a result of non-contracting rule in an even case, and non-expanding rule in an odd case, in a rewriting system form a former section.

The cut rule is of the form

(CUT) $\dfrac{X \Rightarrow Y,\ Y \Rightarrow Z}{X \Rightarrow Z}$.

Let MS denote the system axiomatized by (Id), (LA), (RA), (LIND), (RIND), (BLA), (BRA), (β - LA), (β - RA), (BLIND) and (BRIND). Let MS' denote the system MS enriched additionally with a cut rule (CUT).

9.3.1 Cut elimination for the systems with modalities

We show that for above mentioned systems the following theorems hold:

Theorem 23 *For all types X, Y, $X \Rightarrow Y$ holds in the sense of a rewriting system if and only if $X \Rightarrow Y$ is provable in MS'.*

Proof. Assume $X \Rightarrow Y$ holds in the sense of the rewriting system.Then, there exist types $Z_0, ..., Z_n$, $n \geq 0$, such that $Z_0 = X$, $Z_n = Y$, and $Z_{i-1} \rightarrow Z_i$, $1 \leq i \leq n$. We show that $Z_{i-1} \Rightarrow Z_i$ is provable in MS', for $1 \leq i \leq n$. (Here we show it only for a few chosen cases.)

1. If $Z_{i-1} \rightarrow Z_i$ is the case of (CON), so it is of the form $X, p^{(n)}, p^{(n+1)}, Y \rightarrow X, Y$, we apply (LA) to axiom $X, Y \Rightarrow X, Y$. We get $\dfrac{X, Y \Rightarrow X, Y}{X, p^{(n)}, p^{(n+1)}, Y \Rightarrow X, Y}$.

7. If $Z_{i-1} \rightarrow Z_i$ is the case of (IND), so it is of the form:

7.1. $X, p^{(2n)}, Y \rightarrow X, q^{(2n)}, Y$,for $p \leq q$, we apply (LIND) to axiom $X, q^{(2n)}, Y \Rightarrow X, q^{(2n)}, Y$. We get $\dfrac{X, q^{(2n)}, Y \Rightarrow X, q^{(2n)}, Y}{X, p^{(2n)}, Y \Rightarrow X, q^{(2n)}, Y}$. We can also apply (RIND) to axiom $X, p^{(2n)}, Y \Rightarrow X, p^{(2n)}, Y$. We obtain $\dfrac{X, p^{(2n)}, Y \Rightarrow X, p^{(2n)}, Y}{X, p^{(2n)}, Y \Rightarrow X, q^{(2n)}, Y}$.

7.2. $X, q^{(2n+1)}, Y \rightarrow X, p^{(2n+1)}, Y$, for $p \leq q$,we apply (LIND) to axiom $X, p^{(2n+1)}, Y \Rightarrow X, p^{(2n+1)}, Y$. We get: $\frac{X, p^{(2n+1)}, Y \Rightarrow X, p^{(2n+1)}, Y}{X, q^{(2n+1)}, Y \Rightarrow X, p^{(2n+1)}, Y}$. We can also apply (RIND) to the axiom $X, q^{(2n+1)}, Y \Rightarrow X, q^{(2n+1)}, Y$. We get then: $\frac{X, q^{(2n+1)}, Y \Rightarrow X, q^{(2n+1)}, Y}{X, q^{(2n+1)}, Y \Rightarrow X, p^{(2n+1)}, Y}$.

So, if $n = 0$, then $X \Rightarrow Y$ is an axiom (Id), if $n > 0$, then $X \Rightarrow Y$ is provable in MS', using cut rule (CUT).

Assume that $X \Rightarrow Y$ is provable MS. We show that $X \Rightarrow Y$ holds in the sense of the rewriting system.

If $X \Rightarrow Y$ jest (Id), then the claim is true. For inference rules we show, that if the premise (premises) holds (hold) in the rewriting system, then the conclusion holds in this system. (Again, only a few chosen cases.)

1. For (LA), the antecedent of the conclusion can be transformed into the antecedent of the premise by (CON).

7. For (βLA)the antecedent of the conclusion can be transformed into the antecedent of the premise by (β-CON).

11. For (CUT),if the premises hold in the rewriting system, then the conclusion also holds in this system, since $\Rightarrow$ is transitive. □

Theorem 24 (Cut elimination theorem) *For all types X, Y, $X \Rightarrow Y$ is provable in MS if and only if $X \Rightarrow Y$ is provable in MS'.*

Proof. The 'only if' part is obvious. If for all types X, Y, $X \Rightarrow Y$ is provable in MS (without CUT), it is also provable in MS'.

Assume that $X \Rightarrow Y$ is provable in MS'. By the theorem 23, $X \Rightarrow Y$ holds in the rewriting system. From the theorem 21 there exists such type U, that $X \Rightarrow U$ holds only by using non-expanding rules, whereas $U \Rightarrow Y$ holds only by using non-contracting rules. Thus, there exist types $Z_0, ..., Z_m$, ($m \geq 0$), such that $Z_0 = X$, $Z_m = U$ and for all $1 \leq i \leq m$, $Z_{i-1} \rightarrow Z_i$ is a result of non-expanding rules. We show that $Z_i \Rightarrow U$ is provable in MS, for all $0 \leq i \leq m$. $Z_m \Rightarrow U$ is an axiom (Id). Assume that $Z_i \Rightarrow U$ is provable in MS, $i > 0$. If $Z_{i-1} \rightarrow Z_i$ is (CON), then $Z_{i-1} \Rightarrow U$ is a result of applying (LA) to $Z_i \Rightarrow U$. If $Z_{i-1} \rightarrow Z_i$ is $(B - CON)$, then $Z_{i-1} \Rightarrow U$ is a result of applying (BLA) to $Z_i \Rightarrow U$. If $Z_{i-1} \rightarrow Z_i$ is $(\beta - CON)$, then $Z_{i-1} \Rightarrow U$ is a result of applying (βLA) to $Z_i \Rightarrow U$. If $Z_{i-1} \rightarrow Z_i$ is (IND), then $Z_{i-1} \Rightarrow U$ is a result of application (LIND) to $Z_i \Rightarrow U$. If $Z_{i-1} \rightarrow Z_i$ is $(B - IND_c)$, then $Z_{i-1} \Rightarrow U$ is a result of applying (BLIND) to $Z_i \Rightarrow U$. If $Z_{i-1} \rightarrow Z_i$ is $(B - IND_p)$, then $Z_{i-1} \Rightarrow U$ is a result of applying (BLIND) to $Z_i \Rightarrow U$.

Now, there exist types $V_0, ..., V_n$, $n \geq 0$, such that $V_0 = U$, $V_n = Y$, an for all $1 \leq i \leq n$, $V_{i-1} \rightarrow V_i$ is a result of applying a non-contracting rule. We show that $X \Rightarrow V_i$ is provable in MS, for all $0 \leq i \leq n$. $X \Rightarrow V_0$ is provable in MS from the first part of the proof. Assume that $X \Rightarrow V_{i-1}$ is provable in

MS, $1 \leq i$. If $V_{i-1} \rightarrow V_i$ is (EXP), then $X \Rightarrow V_i$ is a result of applying (RA) to $X \Rightarrow V_{i-1}$. If $V_{i-1} \rightarrow V_i$ is $(B - EXP)$, then $X \Rightarrow V_i$ is a result of applying (BRA) to $X \Rightarrow V_{i-1}$. If $V_{i-1} \rightarrow V_i$ is $(\beta - EXP)$, then $X \Rightarrow V_i$ is a result of applying (βRA) to $X \Rightarrow V_{i-1}$. If $V_{i-1} \rightarrow V_i$ is (IND), then $X \Rightarrow V_i$ is a result of applying (RIND) do $X \Rightarrow V_{i-1}$. If $V_{i-1} \rightarrow V_i$ is $(B - IND_e)$, then $X \Rightarrow V_i$ is a result of applying (BRIND) to $X \Rightarrow V_{i-1}$. If $V_{i-1} \rightarrow V_i$ is $(B - IND_p)$, then $X \Rightarrow V_i$ is a result of applying (BRIND) to $X \Rightarrow V_{i-1}$.

Thus, we showed that $X \Rightarrow Y$ is provable in MS. □

9.4 Conclusion

In this paper we presented pregroups with modalities. First, we presented them in the form of a rewriting system, then we proposed the sequent system for them and finally showed the connections between those two presentations. Using those connections we were able to prove the cut elimination theorem.

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