Surfaces and Superposition

Ernest Adams

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Foreword

In the enormous literature on the foundations of geometry, Ernest Adams' *Surfaces and Superposition* occupies a unique position. Using standard mathematical concepts and theorems from point set topology, as well as psychological theories and results from experiments on perception, Adams presents an extended philosophical analysis of applications of topology to our ordinary experience of surfaces in this insightful work. It is the use of results from mathematics and psychology, while remaining strictly philosophical in method and style, that accounts for this book's remarkable depth.

From another standpoint, the present work constitutes an extended commentary on certain essential features of Euclidean geometry, especially as systematized in Euclid's *Elements*. Using Euclid's controversial concept of the superposition of two figures, Adams—with his natural empirical bent—rightly points out that, although superposition can be avoided in axiomatic developments of geometry, it is an essential aspect of the standard use of measuring instruments in geometry. What he has to say here is just as important and original as his application of topological concepts.

There is much else to be remarked on, but I will restrict myself to another major thread of the book. This is the detailed analysis in Chapter 4 of the concept of physical abstraction and the many interwoven remarks in later chapters on the problems of giving a proper philosophical account of the nature of abstraction. I am not sure I fully understand all aspects of his viewpoint, but I certainly agree with Adams' criticisms of what Russell, Whitehead and some other prominent philosophers have had to say on this difficult subject.

Many readers, not just philosophers, will find much of interest to think about and reflect on in *Surfaces and Superposition*. The unsolved problems that are carefully delineated in numerous passages constitute a worthy challenge to those interested in deepening the foundations of geometry.

I have known Ernest Adams for more than forty years, in fact since we were young men together at Stanford in the 1950s. We are separated by only a few years in age. Almost from the beginning, when he was writing his dissertation and I was his thesis advisor, I really thought of him as a younger colleague rather than a student. He listened intently to comments and suggestions, always holding on to his own independent and original ways of dealing with philosophical ideas.

Over the years he has contributed to a surprisingly wide range of topics in philosophy. His doctoral thesis, a contribution to the philosophy of physics, was on the foundations of rigid body mechanics, remnants of which appear in the chapter on rigid frames and in other parts of the book. Beginning in 1956, a year after he completed his thesis, and over the next decade or so, he wrote a number of papers on utility theory and game theory. Utility theory, especially, is close to the general theory of measurement, and already by the mid 1960s he was publishing in this area as well. Some of his skepticism about overly formal approaches to conceptual problems came out early in his criticisms of representational theories of measurement, and echoes are to be found in the present work. This interest in measurement theory has persisted through the decades and may be found in his 1992 book, *Archaeological Typology and Practical Reality*, on the problems of classification in archeology, written with his brother William.

Still another important strand of Adams' philosophical work began in the 1960s. This is his probabilistic approach to the analysis of conditional sentences in ordinary language, summarized in his 1975 book on the logic of conditionals. It is the body of work, to which he is still contributing, that is probably best known among philosophers.

During the same decade, in 1961, Adams published his first article on geometry whose title, "The Empirical Foundations of Elementary Geometry," already announces the overall theme of the present work. The many long and substantive articles he has published since on the foundations of geometry attest to the permanence of his interest. What is special about the current book is that it has the feeling of a work on which the author has been reflecting for much of his career. The many details and leisurely asides, often as historical footnotes, provide the signs to those of us who know him well that surfaces and superposition engages him at the deepest level.

Patrick Suppes

Preface

The originally planned subtitle of this work, "Field notes on some geometrical excavations," now seems too flippant for so august a production as a book. Nevertheless, an archaeological metaphor is in some ways more fitting for this book's origins, aims, and methods than the 'foundational' or 'architectural' metaphor that is common for studies in the 'conceptual underpinnings' of scientific disciplines. Let us contrast these metaphors, beginning with the architectural one.

There has been a tendency in the past half century to picture theories, and geometrical theories in particular, as formal structures whose superstructures of 'theorems' are based on—deduced from—'primitive' postulates and concepts that are not themselves deduced from or defined in terms of anything else within the structure. The mathematician concerned with deductive structure generally ignores the non-deductive, 'primitive' side of his subject, but the epistemologist seeking justifications for its postulates and the meaning of the concepts in terms of which they are formulated looks for an 'interpretation', total or otherwise, that as it were 'bolts them to a foundation in the world of facts'.

There are important disagreements as to the nature of the foundations, as to the facts and the bolts that tie theories to them, but in most cases they seem to have been pictured somewhat like the flat cement slabs that commonly serve as foundations for small structures. Still metaphorically, they are formed of undifferentiated 'matter', or perhaps Tractarian 'objects' all having the same properties and among which the same relations apply. Moreover, the discovery that Geometry has an empirical element makes it natural to suppose that the facts underlying it have sensory components—sense data, appearances, or the like. Thus, Russell's wonderful "The relation of sense data to physics," in which the 'bolts' that connect the sense data, first to Geometry and then to Physics, are set-theoretical constructions. Reichenbach's 'coordinating definitions' are less radical but nevertheless related efforts to connect geometrical concepts to concrete things, which in his case were rigid rods. In each case there was just one kind of concrete 'fact', part of an 'experiential foundation' secure enough to bear the weight of the theory built upon it, and entities of the theory are bolted to it by connections of specified kinds—constructions in one case and coordinative definitions in the other.

This 'concrete slab picture' was the way I conceived things when I first began these studies, now some 40 years ago. This guided my first efforts to arrive at an understanding of the rôle of Geometry in our dealings with the world, but through repeated efforts to work out the details I came to regard this as misguided, and to adopt a rather different attitude, one that is associated with a different metaphor, namely an archaeological one. This has something in common with the architectural metaphor.

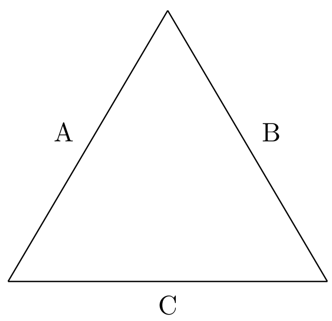
An architectural foundation supports the building that is built on it, but it is *hidden* below or perhaps around the superstructure, which is naked to the eye. And, the fact that the substructure, the 'underpinning', is hidden makes it difficult to map or 'reconstruct'. It must be 'unearthed', as it were, and doing that requires an effort quite unlike that involved in building its superstructure. And here is the important point: one should not begin by assuming that the underpinning is like one kind of common architectural foundation, much less that it has a single level with a well defined outline. Conceivably it could be like the roots of a mountain, that stretch down to an indeterminate depth and outward an indeterminate distance. Or it could be like an archaeological site that has been occupied and built and rebuilt over for centuries or millennia. In any case, the archaeologist must excavate, perhaps layer by layer, perhaps by 'trenching', and he cannot predict with any certainty in advance what he will find. Moreover, what he will find is not likely to have the order and completeness that is found in the buildings that are above ground, since even if the buildings that were there in earlier times had this order, their crumbled remains may only consist of fragments piled up helter-skelter. Then the archaeologist will begin by compiling field notes, noting the locations of the fragments, their forms, probable uses, etc., which may provide the raw materials on which will be based the report that he eventually hopes to write.

My 'conceptual excavations' have not been unlike archaeological ones. I dug and trenched in ways that initially seemed plausible, and found myself sifting through 'conceptual debris' that I thought might be relevant to my inquiry. I might even have proceeded à la Foucault or Derrida, and dug into purely linguistic materials, were it not that a hint from Euclid pointed me in another direction. That is the relation of geometrical concepts and propositions to practical procedures. Thus, Proposition 1 of Book I, which is the first demonstrated proposition in *The Elements*, is "On a given finite straight line to construct an equilateral triangle" (Heath, 1925, p. 241), which sounds and is like a culinary recipe. Moreover, the very concepts in terms of which geometrical propositions are stated are characterized in constructive terms, e.g., parallel lines are defined as " ... lines ... that do not meet *when they are produced* ..." (Definition 23 of Book I). Of course, following Plato, modern mathematicians reject constructibility in the concrete realm and replace it by existence in an abstract or 'ideal' one, but when we seek 'Geometry's application to the world' we may not be ill-advised to examine its technological origins. This seems to me to have enormous implications.

One is that if 'the facts underlying Geometry' have to do with how to do things, then they transcend the static realm of 'eternal existents' to which it is generally thought that its truths should correspond. The other has to do with the things that Geometry instructs us how to construct. What are those? Take the equilateral triangle of Proposition 1. It is something that is drawn on a plane surface. And what can be drawn on a surface but a drawing? It is true that this brings us face to face with the traditional question concerning the function of geometrical drawings. Are they as Plato had it, only 'reminders' of the things that Geometry is really about, which are in a timeless, ideal realm which has no necessary connection to the realm of the senses (and is this really so different from current popular mathematical philosophies)? I think that if there is any truth in this, it is only in a special sense.

I am going to suppose that while geometrical theory can be regarded as being about *idealized* drawings on *idealized* surfaces, its propositions inform us about real drawings on real surfaces in somewhat the same way that the proposition that adding 1 gallon to 1 gallon of water yields 2 gallons of water informs us concerning the result of pouring the contents of a 1 gallon can of water into a container already containing 1 gallon of it, even though neither container contains exactly 1 gallon, and what they contain isn't pure H_2O. Of course, even granting that Geometry can be used to inform us about practical drawings (e.g., Euclid's directions do tell us how to draw real triangles on fairly flat surfaces that are fair approximations of ideal equilateral triangles), the obvious question is: how is it that this discipline plays such an important rôle in modern science? Well, at present that lies close to the horizon in my 'excavations'. What lies more immediately to hand is at a lower and possibly more primitive level: what are real diagrams, which idealized geometrical theory supposedly informs us of? This question is especially acute because our findings suggest that real diagrams are not the 'stuff of physical theory'. But I am going to suggest that being clearer about the 'stuff of diagrams' helps to clarify their relation not only to geometrical theory but to physical theory as well. And, we also find more order than one might expect at this primitive level, that links it first to applied topology, and via that to Geometry. A brief word about that is suitable to this preface, if only as a word of warning.

Take a triangle drawn on a flattish surface, perhaps following the directions given in the demonstration of Proposition 1. A similar triangle appears below, though, significantly, what



actually lies before the reader was printed and not drawn. In any case it is a figure that itself has features: three solid and fairly straight sides, and an inside and an outside. Moreover the sides have *inner* and *outer* edges that are themselves fairly straight, and the edges meet at points, or 'vertices'. Even putting aside the geometrical idea of straightness that this brief inventory includes, note the essentially 'proto-geometrical' ideas it involves: of edges, of insides and outsides, and of edges meeting at points. If the ideas of sides and edges enter seriously into geometrical theory, as they did in *The Elements* (cf. Proposition 7 of Book I), we must examine them in the same way Euclid did the crucial concept of parallelism. And that leads us to Topology, which at least in its origins as *Analysis Situs* (Leibniz, 1956, pp. 254–258) aimed to systematize and examine the logical interconnections among intuitions that underlie our 'ordinary usages' of terms like 'inside', 'outside', 'boundary', and so on. Two points of fundamental importance are connected with this.

One is that an intermediate objective of the theory to be developed here is to 'bolt' topological theory to a foundation in empirical facts and phenomena, beginning with the most fundamental concepts of Topology, that of an *open space*, or an open set. But the distinction between open and closed spaces, or at least open and non-open spaces, is problematic. The most obvious fact is that the distinction is not a metrical one, still less a *geo*metrical one. For example, the distance between the space inside the above triangle, which is naturally regarded as an open space, and the boundary of the triangle, which must be added to it in order to create a closed space, cannot be measured, nor can it be discerned by the eye or any other organ of sense. And yet the distinction is crucial to applied Topology.

This brings us to a rather radical speculation. To make the distinction in the case of the triangle, we imagine how it was drawn or otherwise came into being. For instance, if it was drawn or printed on white paper, then we conceive it to include its boundary points or edges, while its interior is 'background', whose boundary belongs properly to the triangle and not to the interior. On the other hand, if the paper had originally been black, and the white 'interior' had been printed on *that*, it would have included its boundary, and it would have individuated a closed set. Nor is the distinction between foreground and field entirely *ad hoc*, which brings us to the deepest level so far reached in our 'excavation'.

Take the sides of the triangle that are labelled 'A' and 'B'. How do we know that they come together at the top of the triangle; in fact, how do we 'identify' them at all? How do we know that the 'A' and the 'B' label distinct 'things', and are not placed adjacent to just one thing: the whole triangle? Our tentative answer is that we conceive the triangle as having been formed in a particular way, namely by drawing what we have been calling its 'sides' separately. Moreover, these imagined 'acts of construction' not only identify the things constructed, they also determine their relations of incidence, of touching or being separate. The details, some of which are discussed in section 2.3, are messy, but they bring out something of great importance. That is that not only are the fundamental concepts of applied Topology defined 'genetically', but so are the identities of the very objects to which they apply.[1]

Summing up what was said above, three levels have so far been encountered in our excavations, aspects of which will be discussed in what follows. These are: (1) that of *diagrams*, which Geometry provides 'recipes' for creating and about which it provides idealized information; (2) the analysis of topological aspects of the diagrams such as interiors and edges; and (3) 'genetic' characterizations of fundamental topological concepts, which are linked to the very idea of concrete 'thing'. Now, in what follows the discussions of (1) and (3), the diagram and genetic characterization levels, will remain largely at the 'field note' stage. The discussion of level (2), the analysis of topological aspects of diagrams,

[1]The fundamental link between Topology and physical identity should come as no surprise, given the connection of the latter to *continuity*, which is a topological concept.

will be more systematic. It will set forth a deductive theory whereby, starting from intersection relations among parts of diagrams like the sides of the triangle above, one arrives first at *points* of intersection, then at open spaces, and then at boundary concepts, dimension, and linearity, in each case describing 'operational tests' for determining, e.g., that the surfaces on which diagrams are produced are two-dimensional. This is the part of the discussion already mentioned, that aims to 'bolt abstract topological theory to a world of diagrammatic facts'. However, the reader is warned that this 'operational' theory turns out to be anything but 'elementary', and, like topological theory in its early days, it is still only partially worked out. Our excavation is anything but complete, even at its most carefully studied level.

And what about levels as yet hardly touched, including connections with *physical* geometry—i.e., with the geometrical concepts that enter into current physical theories? Obviously these theories are not concerned with diagrams; at best diagrams appear as 'representations' in physical papers and treatises. Well, a first, tentative step in that direction is taken by extending the account of diagrams and figures on the surfaces of bodies to *superpositions* of these bodies that result when the they are fitted together. This in turn leads in two directions: (1) to an account of *spatial measurement* carried out with the use of measuring rods that must be superimposed on objects in order to measure them, and (2) to an account of the *spaces* associated with actual and possible *frameworks* of fitted together bodies. However, simple as the idea of superposition may seem, the fact that superimposing one body on another may conceal the very diagrams that were 'unearthed' at a lower level gives rise to new difficulties not unlike ones that result when one level in an archaeological site has to be disturbed in order to get at a another one. Given this and other difficulties, our discussion of superposition and things that one might hope to account for in terms of it, such as spatial measurement and spaces, remains at the field note level. Still more so possible connections with geometrical concepts of physical theories, discussions of which, except for very cursory comments on the relation of 'superficial signs' to *matter*, are dismissed here as being peculiar to the theories in question.[2]

[2] In any event, given the state of modern Physics, it would seem unthinkable to attempt to account for the geometrical concepts that enter into its theories independently of the conceptions of *light* that they presuppose. And light is a very difficult concept, as one recognizes in the too often forgotten fact that we do not see images on the retinas of our eyes, much less the light that falls on them, and in Einstein's insight that we cannot follow a light signal. Given this and other difficulties, light and its properties are left entirely out of consideration here.

Concluding, I wish to acknowledge the invaluable help and encouragement in carrying out the studies reported here that I have received from individuals and institutions. My greatest debt is to Professor David Shwayder, who, in the early stages of this work, conducted with me seminars on Space and Time, during which views that evolved in one form or another into the *leitmotifs* that inform this study began to take shape. Moreover, Professor Shwayder has over the years rendered me the most valuable help that one writer can offer another, namely that of detailed, careful reading and criticism of the latter's efforts. Chapters 16–21 of Professor Shwayder's *Statement and Referent*, 1992, sketch his own views on topics treated here. More detailed expositions have still to appear, which, together with increasing divergence from my own views, is why these ideas are not discussed in this book. Professor Vann McGee is another with whom I have had profitable discussions ranging over a very wide range of topics, including ones touching on space and time. And, I cannot fail to mention my debt to Professor Patrick Suppes who has helped and encouraged me in almost all my efforts ever since graduate school. I should also like to express my deepest appreciation to Dikran Karagueuzian, the Director of CSLI Publications, as well as members of his staff who helped in the preparation of this book, Christine Sosa, Lauri Kanerva, and Max Etchemendy, who prepared the book's many diagrams.

As to institutional assistance, I am grateful to the Guggenheim Foundation, the Institute for Advanced Study in the Behavioral Sciences, and the National Science Foundation for grants and fellowships in support of the present studies. The present volume is the too long delayed and all too partial recompense for this assistance.

Part I

Preliminaries

1

Characteristics of the Approach: The Case of Points

1.1 Introduction

This study takes some first steps towards the development of a theory of applied Geometry. The goal of such a theory should be to give 'empirical definitions' of the fundamental concepts of Geometry such as "point", "straight line". and so on. However, we will make only limited and probably disappointingly little progress towards that goal, namely to account for applications of the concepts of point set topology to the surfaces of material bodies, and the extension of that to superpositions of these surfaces.

It may seem strange that in order to reach Geometry we should detour by way of point set topology, given that Geometry is generally regarded as an elementary mathematical subject and point set topology uses the full resources of higher-order set theory. But there are compelling reasons for taking this course. In some ways Geometry stands to its foundation as Arithmetic stands to *its* foundation, involving as that does concepts of cardinal numbers and one-one correspondences. The superstructure, 'the mechanics of Arithmetic', is among the first things learned in school while its 'underpinning' in one-one correspondences and cardinal numbers is learned later, if at all (and it is significant that the latter science was developed much later than elementary Arithmetic). So too, 'probing' or 'excavating' the foundation of elementary Geometry uncovers ideas that are less elementary than those of the subject for which they serve as foundation. These include the topological concepts of a boundary, an interior, a line, of dimension, and so on. But it has proved to be an arduous enterprise even to relate these ideas to concrete application, and that, together with ideas relating to superposition, will be our enterprise in this essay.

As the reader may already imagine, the way we undertake this enterprise will be quite unorthodox, at least in relation to recent trends in the foundations of Geometry, and this chapter will comment on differences between it and other works on the foundations of Geometry with which the reader may be familiar. The following section will note certain themes that are characteristic of our approach, and contrast them with more orthodox ideas, and succeeding sections will illustrate them by giving a brief sketch of their application to the case of *points*.

1.2 Characteristics of the Approach

Beyond the topological focus, the most basic difference between our approach to the foundations Geometry and others is that even though our approach is not specifically Euclidean, it follows Euclid in placing the theory of the plane (more exactly, the *surface*) before that of Space and solids.[1] In the present case, however, Space with a capital 'S' never enters the picture, and while bodies and their changing relations do, it is by way of their *possible superpositions* and fittings together. This allows us to treat superposition in a Euclidean way, which is more realistic than Helmholtz's, and which marks our approach's most radical departure from orthodoxy. Let us comment on that at more length.

Euclid has been criticized for giving superposition a rôle in his theory e.g., in the proof of the all-important Proposition 4 of Book I of *The Elements*, that triangles with two sides and included angle are congruent. Thus, Russell wrote:

> The fourth proposition is the first in which Euclid employs the method of superposition—a method which, since he will make any detour to avoid it, he evidently dislikes, and rightly, since it has no logical validity, and strikes every intelligent child as a juggle. (*The Principles of Mathematics*, Second Edition, p. 405.)

and Hilbert and other modern writers on the foundations of Geometry prove essentially the same proposition without reference to superposition (Chapter 11 will return to these points). The author would agree with Russell on this matter, but at the same time insist on the following point made by Euclid's great translator:

[1]It is notable that concepts of space only enter the picture in Books XI–XIII of Euclid's *Elements* (Heath, 1956, Vol. III). As will be noted below, the theory developed in this work is restricted mainly to topological aspects of the foundations of geometry, and its two-dimensional part is compatible with all of the standard two-dimensional geometries.

> In the note on *Common Notion 4* I have already mentioned that Euclid obviously used the method of superposition with reluctance, and I have given, after Veronese for the most part, the reason for holding that the method is not admissible as a *theoretical* means of proving equality, although it may be of use as a *practical test*, and thus may furnish an empirical basis on which to found a postulate. (Heath, 1956, Vol. I, p. 249).

Following Heath, the author would suggest that superposition can be regarded as a fundamental *physical test* of geometrical equality, i.e., of congruence. However, and this is the cardinal point, it cannot serve as the basis of an *empirical* theory if it is interpreted in the way Helmholtz does in his celebrated memoir "On the fact underlying Geometry" (1868), where it is a relation between three dimensional physical bodies and regions of space. That is both because regions of space are not empirical observables,[2], and because things like material bodies that can be observed cannot be superimposed on one another is such a way as to be *totally congruent*, i.e., fill exactly the same space.[3] But no such objection applies to coincidences that arise when *surfaces* are superimposed and concrete things on them are brought together face-to-face, so long as those things are not themselves three dimensional physical bodies. However, the crucial point is that if this is to be possible it requires us to recognize the existence of certain concrete observables in the external world that are not physical bodies—certain 'quiddities' that transcend the traditional empiricist mind-body dualism.

Concreta that can fill this rôle are what we will call *surface features*, examples of which include bumps, dents, scratches, sticky spots, and, preeminently, visible marks and figures like the letter 'S' and the hollow triangle, '⊻'. These are undeniably publicly observable things in the external world; in fact, they are on the page's surface, although they are not parts of it. That they might be *material*, or collections of material particles, and therefore be three dimensional, will be considered and rejected in section 2.2. But the point to make here is that the properties

[2]That is why we cannot accept Whitehead's theory of points as arrived at by processes of extensive abstraction, starting with spatial regions. Analogous objections apply to mereological theories of spatial 'structure', at least so long as they take the 'parts' that they deal with to be parts *of* something like three-dimensional continua.

[3]The same point was made by Rudolf Hertz in commenting on Helmholtz's paper, cf. Helmholtz, 1868, Lowe translation, in Cohen and Elkana, 1977, pp. 43 and 62. But as stated, Chapter 4 will characterize the *physical abstraction relations* that subsist between certain physical observables and abstract spacelike regions with which they coincide.

and relations of surface features like these will be the foundation on which the theory of surface topology to be developed in Parts II and III of this work rests. As such they will be of fundamental importance for us, and all of Chapter 2 will be devoted to this metaphysically unfamiliar kind of 'thing', which, because it has no place in the world outlook of modern science, the reader may be uninclined to take seriously. But it is hoped that she or he will at least accord them provisional acceptance, if only to see what this may explain.

Let us then note certain immediate consequences of taking them to be the fundamental objects of geometrical application.

One is that Geometry is not treated as the science of space á la Newton, Kant, and innumerable others, and we are led to a relativism that is even more radical than that of Leibniz. Moreover, the fact that we treat Geometry as a science of concrete, two-dimensional, and immaterial objects implies that we do not treat it as a branch of Physics. In fact, the problem of characterizing the geometrical concepts that are presupposed in the formulations of this or that physical theory, Newtonian, Einsteinian, or whatever, is regarded as peculiar to that theory.[4]

More generally, not only is our theory non-physical, but we do not regard Geometry as empirical in the ordinary sense at all. Although we hold that the primary concrete observables that Geometry *applies to* are two dimensional, non-material features of bodies' surfaces,[5] we will also hold that those are not geometrical objects 'proper', i.e., they are not what the variables of geometrical theory range over. Rather, the relation between geometrical entities and these concrete observables will be held to be one of *abstraction*, which is more like the relation between *universals* and the particulars that fall under them.[6] In a way the relation of Geometry to application is better modeled on that of Arithmetic to the concrete, as analyzed in Cantor's work, where natural numbers are arrived at by a process of abstraction, starting with concrete objects and moving first to classes that have those objects as members,

[4]The reader may reasonably ask what relevance this study might have to the philosopher of Physics who seeks to illuminate the geometrical presuppositions of the theories that concern her or him. This will be returned to briefly and inconclusively in Chapter 16, but here it may be said that the present study seeks to make explicit and analyze things that are part of our common heritage, which are presumably assumed, possibly unconsciously, in all geometrical thinking, including that of physicists.

[5]To be sure, it will be held that geometrical entities like points stand in well defined relations to objects that can be discriminated by the senses, but it will not be held that they *are* such objects, or that they are *physical* in the ordinary sense.

[6]It follows that on this view, while Geometry can be applied, and accounting for its applications is one of our principle objectives, applied Geometry is not an 'interpreted formal system' after the fashion of Nagel, 1961, Chapter 8, and others.

but which are not themselves objects in the ordinary sense.

The 'Cantorian model' of the relation between abstract mathematical entities—classes in particular—and concrete particulars is significant both for the similarity of this relation to the relation between abstract geometrical entities and concrete *'sensibilia'*, and for important dissimilarities. Perhaps the most important similarity resides in the existential presuppositions of theories of classes and theories of geometrical entities. That entities having the properties affirmed of classes in the theory of classes derives from the use in everyday speech of expressions like "the class of persons born in the United States between 1880 and 1890". Usually users of these expressions do not ask whether there exist *entities* answering to such descriptions, but it is plausible that the ready acceptance of Cantor's Postulate of Abstraction depends on the ubiquitousness of this usage. And, while there is a sense in which this makes the postulate *a priori*, its grounds lie in the nature of our linguistic, rather than in that of our perceptual apparatus.

As to geometrical theory, we shall maintain a similar thesis with respect to certain postulates of existence that are presupposed in it. For example, we are accustomed to saying things like "He returned the book to its place on the shelf" without questioning the existence of an entity denoted by "its place on the shelf," but it can be argued that the ubiquitousness in ordinary speech of usages like this underpins the sort of abstraction that is fundamental to Geometry. That is what gives certain principles of Geometry an *a priori* character, which we will argue is also synthetic—although not Kantian.

Another similarity between the set-theoretical and the Geometrical cases is related to the fact that while the ready acceptance of the postulates of Cantor's theory derives from their underpinning in ordinary usage, Cantor *regimented* that usage for his purposes, specifically for applications to mathematics. For example, he assumed the atemporality and extensionality of classes, though neither is presupposed in everyday usage. Thus, modern set theory does not envisage the possibility that a thing might belong to a class at one time but not at another, in spite of the fact that it makes good sense in everyday usage to say things like "Jones joined the class of millionaires in 1990." And, in spite of the fact that the classes of unicorns and of golden mountains are both empty, ordinary usage does not recognize them as identical. As to Geometry, we will suggest that its existential presuppositions also regiment ordinary usage, but in a way that differs from that of the theory of classes.

As said, in our theory geometrical entities will be treated as being like abstract *universals*, whose extensions at any one time are Cantorian classes, more than they are like the classes that are their extensions.

That points can be regarded as universals arises from the fact that they are what concrete objects that *meet* in them have in common. In this respect they are like abstract *colors*, which are what objects having those colors have in common. Furthermore, universals like colors have both temporal and modal aspects. An object can change color over time, and an abstract color can exist even if nothing has it, if something *could* have it (think of Hume's imagined shade of blue). Similarly, something can be at a point at one time but not at another, and there may be points in space where nothing is; i.e., our spaces are *container spaces*.

However, our theory of geometrical entities like points will introduce modality in a special way, namely by way of *postulates of constructability* that are closely akin to Euclidean constructability postulates, e.g., that it is possible to draw a straight line from any point to any point (Postulate 1 of Book I of *The Elements*, Heath, 1956, Vol. I, p. 195). Note that this is possibility in *this* world, and our theory of it will be modeled more closely on Euclid's usages than it is on widely known possible worlds characterizations.

Another important aspect of constructive modality brings us back to our primary reason for approaching the foundation of Geometry in a way that allows us to deal realistically with superposition. The most important use of constructions in Euclid's theory arises in giving *operational tests* for geometrical properties, perhaps the most famous of which is the test for the parallelism of straight lines in the plane, i.e., they are defined as lines that do not meet if prolonged (Definition 23 of Euclid's *Elements*, Heath, *op. cit*, p. 154). Thus, we do not tell simply by looking at them that the lines in the figure '−|' are not parallel; we *demonstrate* this by prolonging them, say to form the figure '−|−', in which the extended lines do intersect. Similarly, in the proof of the 'infamous' Proposition 4 of Book I (Heath, 1956, Vol. I, p. 247), two triangles are not proved to be congruent by simply inspecting them, but rather by superimposing one on the other to see whether they coincide. And it is important that this operation has an empirical side: something has to be seen, but an active test must be carried out to make that possible.

Finally, returning to the topological theme, we will be concerned with topological aspects of Euclid's theory, e.g., as in Definition 3 of Book I of *The Elements*, "The extremities of a line are points" (Heath, 1956, Vol. I, p. 165). With caution, extremities may be equated to *boundaries* in the topological sense, but which Euclid left unanalyzed because he lacked the set-theoretical machinery needed to characterize them in a precise way. Combined with set-theoretical machinery, which we will use freely, our constructive approach allows us to do this in the case of the topologies of surfaces, which also leads to characterizations of other topological concepts such as those of an *interior*, of *connectedness* (or continuity), of *dimension*, and of *linearity*— though not of *straightness*, which is a metric concept. While metrical concepts are discussed at some length in Chapters 14 and 15, they are not fully dealt with in this work.[7]

Below we will illustrate the foregoing themes in their application to *points*, but let us summarize them first.

First and foremost, because the two-dimensionality of bodies' surfaces makes it possible to characterize their superpositions in a realistic way, we follow Euclid in taking the surface, rather than the three dimensional *space*, as the point of departure; in fact we never arrive at a single Space of the kind that philosophies of Physics often seek to characterize. It follows that the concrete objects of our theory are not three-dimensional, hence they are not physical in an ordinary sense, and therefore Geometry, construed as a theory of such objects, cannot be a branch of Physics. Second, even though applied Geometry may have empirical content it is not an ordinary empirical theory, and it is not even an interpreted formal system. Rather, geometrical entities like points are related to concrete observables by *abstraction*, somewhat as Cantor's classes are related to the objects that are their members. However, they are even more like *universals* of which classes are extensions than they are like classes, in that they are temporally variable and non-extensional. This non-extensionality is related to *constructability* of the kind that enters into Euclid's theory, rather than being a possible-worlds concept, and an empiricist-operationalist program involving it will be returned to in Chapter 3.

Now we will illustrate the above themes by sketching their application to the case of *points on surfaces.*

1.3 Illustrations in the Case of Points on Surfaces

We see that the dot "•" is fairly small, but it is not as small as a single point; in fact if it were that small it would be too small to be seen. We also see that the three segments composing the figure  do not meet in any point while those composing the figure  do. But how do we account for these things, e.g., for seeing that the segments composing the  have in common something that is too small to be seen?[8] For answer, we take

[7]It will be argued in Chapter 15 that while topologies of bodies' surfaces and even of the matter that forms them can be characterized along lines developed in this work, topologies of *spaces* cannot be characterized in a similar way.

[8]The fruitlessness of Euclid's and others' attempts to define *point*, e.g., as "that which has no part" and as the extremity of a line (Definitions 1 and 3 of Book I of *The Elements*, Heath, 1956, Vol. I, p. 153) is suggestive of the difficulties involved in this concept. Aristotle seems closest to the mark in identifying points with locations

note of two things: (1) the distinction, emphasized by certain ordinary language theorists, between *seeing* and *seeing that*, and (2) the analogy between seeing that two segments like those in the figure −| are not parallel and seeing that the segments in the figure ⧖ do not have a point in common.

In the case of the figure −|, visual observation of the segments forming it leads to an intellectual conclusion: that carrying out the Euclidean test for parallelism would establish that the segments are not parallel. If the horizontal segment were extended it would meet the vertical one, as in −+. This suggests that we should look for analogous sensory and intellectual components involved in 'seeing that' the segments forming ⧖ do not meet in a common point. But what *test* might establish this: that the three segments do not meet in a common point? Euclid did not describe a test that would be analogous to his test for non-parallelism,[9] but we will now describe one, whose properties will be examined at length in Part II.

Imagine the figure ⧖ enlarged as in Figure 1A below:

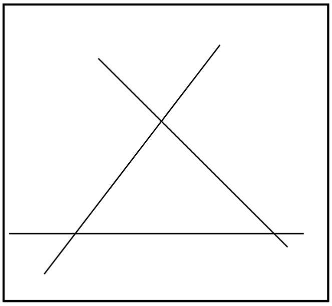


Figure 1A

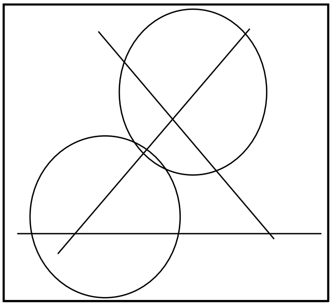


Figure 1B

We still see that the segments in Figure 1A are not coincident, but a test like the one whose result is pictured in Figure 1B would *demonstrate* this. It pictures two 'auxiliary ovals' drawn over the left-hand segment, which *cover* it in the sense that anything touching the segment would

("we can make no distinction between a point and the *place* (*τόποσ*) where it is" (*Physics*, IV. I, 299 a 30, quoted from Heath, *op. cit*, p. 156), but of course *location* must still be analyzed. Footnote 9 comments further on this matter.

[9]It must not be thought that devising such tests is easy. Some writers have regarded Euclid's test for parallelism (Definition 23 of Book I, Heath, *op. cit*, pp. 191–4), and the associated Postulate 5 (Heath, *op. cit.*, pp. 202–20), as being among his most important original contributions to geometrical theory. Heath's discussion makes it clear that it is by no means self-evident what 'the right test' for parallelism is.

have to touch one of them,[10] while *neither oval touches both of the other segments*. Given this, there could not be a point that is common to all three segments. If such a point existed it would be a point of the left-hand segment, and therefore it would be a point of one of the ovals that cover it. But then it couldn't be a point of both of the other segments, since neither oval touches both of them. On the other hand, if such a 'separation test' were applied to an enlargement of the figure , at least one of the ovals covering the left slanting segment would have to touch both of the other segments. This would prove the coincidence of the segments in the same way that failure of the two segments to meet when extended would demonstrate their parallelism.

There is another important point to note about the separation test. Because it is complicated it is not self-evident that it really does provide necessary and sufficient conditions for coincidence. Therefore it is an important part of the theory of coincidence to *justify* this test and ones like it, in the present case by proving that two or more segments meet in a common point if and only if none of them can be covered with figures like the ovals in Figure 1B, in such a way that no one of the covering figures touches all of the given segments. The proofs of many of Euclid's Propositions have the same function, i.e., of showing that the constructive methods described in these propositions have the properties required of them, e.g., showing that a triangle constructed by such and such a method *is* equilateral, which is what the proof of Proposition I of Book I does (Heath, *op.cit.*, p. 241).

Another task of the theory is to explore interconnections between tests for different things. For example, one way of showing that the dot "•" isn't as small as a single point is to draw two lines, each of which touches it, but which don't touch each other, as in the figure . If the dot were as small as a point and both of the vertical segments touched it, they would have a point in common and therefore they would have to touch each other. But here we have a second test involving points, and we would like to know what its relation is to the first one, which seems to be utterly different. The theory of coincidence seeks to demonstrate interconnections between such tests, e.g., that any mark that touched all of the segments forming the figure could also be touched by lines that didn't touch each other, as in the figure .

Another thing that the theory of coincidence seeks to do is explain various intuitions, e.g., that points might be said to have no parts, or

[10]Section 2.4 points out that the *interiors* of regions outlined by visible ovals are in a sense 'secondary features', and 'observing' that they stand in topological relations, like covering, to other visible things requires special analysis. But this will be put off for now.

that anything large enough to be seen is too large to be at a single point—although small dots can 'approximate' and 'represent' them for this reason, which is connected to the fact that it is *hard* to draw separate lines that both touch a dot like "•".

Various other aspects of a theory of points can be noted briefly. One is that it involves *postulates of constructability* that are analogous in many respects to Postulates 1–3 of Book I of Euclid's *Elements*, e.g. that it is possible to draw a line from any point to any point. This postulate guarantees the possibility of carrying out an operation that is involved in many of the proofs in Geometrical theory, e.g., the proof of Propostion 5 of Book 1 (that the base angles of isosceles triangles are equal, and so are their complementary angles; Heath, *op. cit.*, p. 251). An analogous postulate in the theory of coincidence is needed to guarantee the possibility of drawing the kinds of ovals or similar figures that are involved in the separation test described above, which demonstrates the non-coincidence of lines or other figures. In fact, a general *Separation Postulate* will be stated as basic principle 6.3.1 in Chapter 6, where the theory of coincidence is developed systematically.[11]

Another aspect of the theory has to do with the *justification* for affirming the existence of abstract points and other geometrical entities, especially when they are too small or too 'thin' to be seen. What justifies our saying that the segments forming either of the figures + or $\times$ have points in common? It is here that we follow Cantor's example and simply *postulate* the existence of these entities, thus making explicit what is presupposed in the ordinary use of expressions like "the point of intersection of the segments."[12] The postulate is a principle of 'classical physical abstraction' (Adams, 1993), which is *a priori* in the sense that it derives from pre-existing usage, and it is synthetic because it depends in part on the Separation Postulate.[13]

As said, the following chapter will discuss these matters in more

[11] It is to be noted that this postulate only requires the possibility of 'freehand' drawings, in contrast to 'ideal' geometrical figures, which is in keeping with our topological, 'pre-geometrical' approach.

[12] The most direct geometrical analogue to Cantor's axiom would be to postulate the existence of a *place* at which any thing is at any time, where the *exact place* of the thing is appropriately taken to be coextensive with it. Including points in the category of places (or at least of *locations*) is a further step, since they are too small to be the exact places of any concrete things. Footnote 8 suggests that Aristotle may have done this, and section 4.6 will comment on extending the range of the abstract *beyond* what is instantiated in the concrete.

[13] By contrast, so far as I am aware the synthetic aspect of Kant's *a priori* had to do with the qualities of 'phenomena', and nothing to do with what can or cannot be constructed in the real, 'noumenal' world. But perhaps Kant would have held that what we are calling surface features *are* in the phenomenal world.

detail, but let us turn to other matters first.

1.4 Relevance to Geometry

The reader may wonder what the foregoing considerations and the theory to be built on them have to do with the science of Geometry. Since Plato we have been taught that diagrams like Figures 1a and 1b are mere 'visual representations' of real geometrical objects. Thus, in the *Meno* Plato held that Geometry is not concerned with visual diagrams, or as Wedberg put it, he held that "There are no truly Euclidean objects in the sensible world" (Wedberg, 1955, p. 49). More generally, Plato held that

> ... no one who has even a slight acquaintance with geometry will deny that the nature of this science is in flat contradiction with the absurd language used by mathematicians ... They constantly talk of 'operations' like 'squaring', 'applying', 'adding', and so on, as though the object were to *do* something, whereas the true purpose of the whole subject is knowledge—knowledge, moreover, of what eternally exists, not of anything that comes to be this or that at some time ... (Chapter XXVI, §2, of *The Republic*, p. 244 of the Cornford translation).

Of course, mathematicians no longer use 'absurd words' like 'apply' in the way that Euclid used them, e.g., in the proof of Proposition 4 of Book I of *The Elements*, which states that triangles are congruent if two sides and their included angles are equal (Heath, *op. cit.*, p. 247).[14] But the theory developed here will be non-Platonic, and it will be supposed to apply to visible figures, which are not supposed to be mere representations of something *else*.[15]

But even supposing that *our* theory applies literally to Figures 1a and 1b and others like them, it isn't clear that they are objects of *geometrical* theory. We have already set aside the geometrical aspects of theories of Physics (e.g., Newtonian or relativistic mechanics) as being special to those theories, but are these not *the* empirical applications of Geometry?

[14]Note that Hilbert took substantially this proposition without proof, as an axiom, namely Axiom IV.6, on p. 15 of the Townsend translation of *The Foundations of Geometry*, 1902. But as will be pointed out in section 11.1, even as late as the middle of the last century, geometry texts like Legendre's celebrated *Éléments de Géométrie* continued to speak of 'applying' figures.

[15]In fact, in some ways our approach fits in better with Berkeley's characterization of abstract ideas as particulars that are 'rendered universal' by the fact that we suppose that certain things that are true of them are true of all figures like them (cf. section 15 of the Introduction to *On the Principles of Human Knowledge*).

It might seem that whatever is left over after those aspects have been set aside are things of naive, everyday speech such as what might be referred to as 'the edge of the field', to which the rules of thumb of carpenters, surveyors, and others unversed in modern physical theory (and perhaps classical geometry as well) apply. But that would be too simple.

Even in ancient Greece persons applying Geometry had little use for theory. As Plato said "...for such purposes a small amount of geometry and arithmetic will be enough" (p. 243 of Cornford translation of *The Republic*; the purposes that Plato referred to were applications in 'warlike operations'). But he did hint that theory is useful in attaining exactitude. For instance, commenting on the place of astronomy in the education of the Guardians, he wrote:

> ...we must use the ... heavens as a *model* to illustrate our study of those realities, just as one might use diagrams exquisitely drawn by some artist like Daedalus. An expert in geometry, meeting with such designs, would admire their finished workmanship, but he would think it absurd to study them in all earnest with the expectation of finding in their proportions the *exact ratio* of any one number to another. (p. 248 of Cornford translation, my italics).[16]

Exactitude is suggestive. Actual diagrams are not drawn by ideal artists and they do not exactly conform to Euclidean specifications. Nor did Euclid attempt to explain how the theory developed in *The Elements* applies to things that don't exactly conform to these specifications. Applying the theory was largely a matter of following rules of thumb, and to some extent this is still the case. But the theory developed in the present essay aims to explain these applications in part, as the example of geometrical points illustrates. Thus, it explains why small dots are not geometrical points, but the smaller they are the closer they come to instantiating points because they come closer to satisfying the requirement that any two things that touch them must touch each other.

In some ways *lines* furnish a better example. Again, no observable thing, even a two-dimensional feature on a surface, is thin enough to be a geometrical line, but it follows from the theory developed in Part II, especially in Chapter 10, that the *edges* of observable things like the long rectangle below are one-dimensional:

[16]The 'realities' referred to were the *general* laws of motion, both celestial and terrestrial, which were little understood in Plato's time. The author's article "Idealization in applied first-order logic" (Adams, 1999) discusses a 'Platonic model' of the relation between inexact empirical theories and the idealized models in terms of which they are often formalized in first-order logic.

Chapter 10 proves this by developing a Platonic conception of an ‘ideal line’ that a concrete thing may ‘partake of’, which we characterize here as being topologically *homeomorphic* to the ideal line. The theory developed in Part III even goes some way beyond this, in the direction of characterizing *straightedges and straightness* (cf. section 14.2), which is a part of the theory of *superposition.* This is fundamental to *Euclidean metrology*, and, concluding this section, we will make two comments concerning its relations both to Geometry and to a branch of modern science.

To superimpose one figure on another is essentially to ‘apply’ one to the other in the sense intended, for instance, in the proof mentioned above of Proposition 4 of Book I of *The Elements.* Analyzing this operation is a first step towards a full-scale analysis of the metrical concepts of Geometry, and while this essay only goes a short way towards doing this, it does solve certain ‘philosophical’ problems that exercised classical thinkers, which are beginning to receive attention once again. These concern the idea of bodies having *common boundaries* or *faces.* Ancient Greek philosophers, e.g., Sextus Empiricus (cf. Mates, 199., p.) as well as modern writers (e.g., Stroll, 1988, Zimmerman, 1996) have puzzled over the question: when two physical bodies are superimposed face-to-face, do the faces that were separate before they were superimposed become fused into one? If they do then the whole bodies ‘become one’ by fusing into a single body, while if they do not fuse then they must still be separate, and therefore they haven’t really been superimposed. The solutions that are offered to this and related problems in Part III are given in terms of *composite surface topologies* which are formed from the surface topologies of their components when they are superimposed. This is too complicated to describe at this point, but the essential point is that the faces that were separate originally, and the features on them, *retain their identities* when they are superimposed, but the topological *spaces* that are generated as a result of their being superimposed are somewhat complicated *extensions* of the component spaces, which are described in detail in Chapter 12.

Superposition is important for another reason. Theories of visual perception, like those of Locke or of Marr, 1982, tend to suppose that visual images ‘resemble’ surface features such as the long rectangle pictured above, and these resemblances provide ‘our knowledge of the external world’. Typical of the problems that arise in this approach is that of explaining how, if all we are ‘given’ are two-dimensional visual images, we arrive at the idea that ‘the world’ is three-dimensional. The fact is that what we perceive by sight *are* features on surfaces, like the long rectangle, but 'transferring' them to mental images leaves out something crucial. Surfaces and their features can be superimposed, and the facts of superposition are what actually inform us both of spatial dimensionality and of metrical facts, but 'superpositional facts' are not transferred to mental images. This is discussed in Chapter 17, which is the penultimate chapter of this essay, and it suggests that visual perception theorists should pay much closer attention to superposition than they have heretofore.

A final point about our overall program is as follows.

1.5 An Empiricist-Operationalist Program

As said earlier, *The Elements* develops a largely constructive-operational program that defines and analyzes interconnections between concepts of elementary geometry. For instance, Definition 23 of Book I characterizes parallelism as the property of straight lines of not meeting when they are extended or 'produced' (Heath, 1956, Vol. I, p. 165), and Proposition 1 of Book I describes a method for constructing an equilateral triangle on a given base (Heath, *op. cit.*, p. 241). The present work applies the operational approach to the topological concepts that underpin geometry, e.g., that of an *extremity*, which enters in Definition 3 of Book I of *The Elements*, namely "The extremities of a line are points" (Heath, *op. cit.,* p. 153).

The present work seeks to give operational definitions that link topological concepts directly to empirical observables, and specifically to the surface features that were discussed in section 1.2. Hence our program may be said to be 'operational', and it is akin in spirit to P.C. Bridgman's old program for analyzing concepts of Physics.[17] This has already been illustrated in the case of the concept of *coincidence*, as applied to the figures  and , in the first of which the segments meet in a point, and in the second of which they do not. Figure 1B pictured the result of carrying out an operation in which one of the segments of  is covered by ovals, the upshot of which is to prove that the three segments are not coincident. This procedure is operational and the ovals that are produced are surface features that are observed by sight, hence it fits into an empirico-operationalist program. This work will extend the method to more properly topological concepts, including ones mentioned earlier, of a boundary, continuity, dimension, and linearity, as well as the most

[17] Cf., "...the concept is synonymous with the corresponding set of operations. If the concept is physical, as of length, the operations are actual physical operations, namely those by which length is measured ..." (*The Logic of Modern Physics*, 1927, p. 5).

fundamental topological concept, beyond that of a point, namely that of an *open set*. Both the operational definitions and their justifications and interrelations are complicated, and details will be set forth at length in Part II, which is inspired by if not modeled on the pattern of Euclid's *Elements*.

But we must note a serious limitation.

1.6 The Problem of Appearance and Reality

The triangle in Figure 1a appears to divide the page on which it appears into an 'inside' and an 'outside', and the constructive *proof* of this is to verify that any continuous line drawn from one side to the other must cross at least one of the triangle's sides.[18] Now, a doubtful reader *could* try to verify the impossibility of doing this by actually trying to draw such a line, but would failure to do so after repeated efforts really prove that that would be impossible? Failure might only show that the reader was too unskillful, or too clumsy, or be using drawing instruments that are too crude. Can we ever *really* prove a physical impossibility—that something can't be done? For answer, we will proceed as follows.

What we will take for possibility is the *appearance of possibility*. Our 'datum' will be the fact that it *appears* to be impossible to draw a continuous line from the inside to the outside of the triangle in Figure 1a, and our theory will concern interconnections between these appearances. That is what will give it, and we would argue give classical geometry as well, its empirical aspect.

But the appearance is itself an idealization that may not correspond to reality, and this is closely connected with another kind of idealization. Although it looks as though it would be impossible to draw a continuous line from the inside to the outside of the triangle, closer inspection—say under a magnifying glass—might reveal that the lines forming the triangle's sides were nothing but closely packed dots—'pixels'. Still closer inspection would certainly reveal that the dots themselves were assemblages of smaller particles whose light-reflecting properties created the appearance of continuity. In any case, though, it *should* be possible to 'thread ones' way' among the particles from inside to outside in such a way as to avoid touching any of them. The 'reality' would conflict with the appearance, and how are we to deal with that? The answer is to stick resolutely with the appearance, while admitting that to speak of 'the' appearance is a gross oversimplification. And, we would suggest

[18] This topological version of Pasch's Axiom can be regarded as a special case of the Jordan Curve Theorem (Courant and Robbins, 1941, pp. 244–246). It is also closely related to the test for a one surface feature to cover the boundary of another that is described in Theorem 8.2.2 of section 8.2 of the present essay.

that the practical application of classical geometrical theory also deals with simplified appearances, e.g., that it should be possible to draw a straight line from any point to any point. These and similar claims are idealizations, but the very fact that they are about appearances is part of what gives geometrical theory an empirical aspect. However, we can always ask: what is their relation to *reality*?

This question has plagued philosophers and philosophically inclined physicists from ancient times to the present. Plato held that the objects of science, and especially of geometry, are accessible only to thought, and not accessible to sensory observation. Descartes held that the senses are only guaranteed to yield veridical information about the external world when they consist of clear and distinct ideas, whose validity is guaranteed by a benevolent deity whose own existence must be inferred by pure reason. Leibniz held that geometry is not the science of absolute Space, but rather of relations between material entities (¶ 62 of the Fifth Letter to Clarke), but he neglected their surfaces. Berkeley and others following him (e.g., Reid, Mach) hold that the proper entities of science are 'ideas' that are directly accessible to observation, but are not material. And, Kant held that the geometrical content of our ideas conforms to laws of our own perceptual apparatus that have no counterpart in the external world. None of these views accords a place to the 'appearances' that concern us here, of bodies' surfaces and their features, so it appears that learned opinion is against us. On the other hand, the diversity of these opinions suggests that the door may be open to other approaches, and in particular to one that is inspired by the status of *diagrams* like Figure 1a, or those that appear in the planar books of Euclid's *Elements.*

But of course, treating diagrams as appearances does not resolve the old appearance and reality problems that Plato *et. al.* attempted to solve. Appearances *are* appearances, and how can we claim that surface features like Figure 1a are publicly observable if they only appear to certain people under certain circumstances? How can we claim that they are 'out in the external world', in the same places as bodies are, if they are mere appearances and bodies are 'really there'? To hold this seems to start down Berkeley's slippery slope towards holding that appearances are all that we have knowledge of, and bodies are mere congeries of them.

Well, any epistemologist knows that these are tremendous questions, and the author does not pretend to be an epistemologist. He can only forthrightly admit that these difficulties lie on the road ahead. But that is far ahead, and they will only be reverted to in the last two chapters of this essay. In the meantime we will simply ignore them, and pretend that we can all 'see' Figure 1, and that it lies on a particular page of this essay.

The following chapters turn to details, but let us first briefly summarize the themes to be dealt with in them.

1.7 Summary of Themes of Following Chapters

The remaining chapters of Part I will develop the ideas just discussed in more detail, but still informally. Part II presents a deductive theory of the topologies of the surfaces of bodies, which defines abstract *points* and *open sets* in these spaces in such a way as to ‘coordinate’ them with concrete observables on these surfaces, and in terms of this it argues for the surfaces’ two-dimensionality, the one-dimensionality of the observables’ boundaries, and other intuitive surface properties.

Part III extends the discussion to consider properties of the ‘composite topologies’ that result when one surface is superimposed on another, or several surfaces are ‘fitted together’. Classical paradoxes of superposition like the ‘fusion problem’ described above are resolved, and new concepts like *orientability* are brought into the picture. A static and relational concept of rigidity is introduced, that holds between bodies if they can only be fitted together in ‘minimum’ numbers of ways. Although the development is informal and not worked out in detail, this suggests a way of building on the theory developed in Part I, to define topologies of ‘frameworks’ consisting of bodies that are rigidly fitted together.

Part IV consists of unsystematic remarks on two further ramifications of the present investigation. Chapter 16 is concerned with relations between the present theory and modern physical theory, which are difficult to define because of the fact that even Newtonian physics deals with material bodies in ‘absolute space’, whereas the present investigation is primarily concerned with non-material features of bodies’ surfaces, and certain relative spaces that are defined by rigid frameworks that these features define. Nevertheless, it is suggested that there are at least two important connections between these realms. One has to do with *measurement*, which is essential to the application of physical theory, but which at its simplest must be carried out by ‘applying’—superimposing—measuring rods to the bodies with which physical theory is concerned. The other has to do with *matter*, which is the ‘substrate’ of which bodies are formed, but which can be ‘recognized within the bodies’ by the kinds of marks that appear on their surfaces.

Chapter 17 discusses very briefly connections between the present theory and certain *theories of perception*, both in philosophy and psychology, that derive from the fact that the visual images that figure in these theories closely resemble surface features like the star ‘⚹’, which we argue are ‘in the external world’. In fact, we very tentatively suggest that adding *non-material* but external and publicly observable surface features to traditional empiricism's mind-body dualism may contribute to solving skeptical difficulties associated with this dualism. Chapter 18, the final chapter, gives a still briefer résumé of the aims and claims of this work, followed by inconclusive comments on six important objections to its principal themes.

Concluding, it must be emphasized that the only systematic, in fact deductively organized, part of this work is Part II, which describes an 'empirico-operational foundation' for an application of present day mathematical topology (especially point-set topology) to the surfaces of material bodies. Part I deals informally and unsystematically with various '*leitmotifs'* that guide the deductive theory in Part II, especially its foundation in an unfamiliar ontological category of 'objects', namely two-dimensional and non-material 'features' of the surfaces of bodies, and principles of abstraction analogous to those of Cantor's theory of sets, which are presupposed in moving from the level of concrete surface features to abstract entities, especially points, that are instantiated in them. Part III is a largely informal sketch of how the theory developed in Part II might be extended to describe the 'composite surface topologies' that are formed when the surfaces of two or more bodies are superimposed, which, arguably, makes it possible to account for metrical concepts of 'geometry proper', such as length and distance. Part IV consists of still more informal reflections on 'applications and implications' of the ideas developed in Parts I–III for modern science, specifically Physics and theories of perception in Psychology. These remarks can at most provide 'food for thought' to students whose primary interests are in these subjects, and the author can only say in his own defense for including them is that these are the topics in which he was originally interested, and it was by prolonged reflection on them that he was led to the views expounded here.